The Inter-industry Propagation of Technical Change

Formulation of a dynamic price system and its application to a stochasitc differential equation

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Our intsitute is named the Keio Economic Observatory, even though an observatory usually means an astronomical or meteological institute for the observation of natural phenomenon. We call our institute an observatory because we wish to treat economics as an empirical science and thereby intend to analyze economic phenomena objectively, being completely detached from any ideologies, by making use of economic theory as an equivalent to theories of other physical empirical sciences. The K. E. O. monograph series, of which this book is one, is designed to publicly demonstrate this spirit. We hope that this book presents a tangible example of economics as an empirical science.

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Introduction

Very rapid price decrease of semi-conductor and of its products such as PC has been common phenomena in recent years.[1] These products are capital goods, because they are durable and used in the industry sectors. The price of capital goods has not only instantaneous effects on the economy, but also has persistent effects through the installed equipment and the opportunity costs.

Traditional neoclassical growth theory assumes the effects of technical change on an economic system are stable residuals, so that productivity growth is both a source of economic development and a stabilizing influence on inflation. However, rapid technical progress such as the one in the microcomputer industry will rapidly increase user cost of installed equipment. Given the accounts of many types of the accumulated capital goods, the effect of technical change may thus become a disturbing factor to a dynamic economic system. This research examines these effects of technical change in a very simple neoclassical growth model that incorporates the accumulation of capital goods.

In general equilibrium analysis, productivity growth or a wage increase in one sector changes not only the own sector's output price, but also affects the output prices in other sectors, because some outputs are inputs to the other sectors. Furthermore, there are commodities that are accumulated as capital goods. For the former static case, many analyses of the inter–industry effects of factor prices have been elaborated in various contexts.[2] But for the latter dynamic process with capital formation, only a few studies have been executed.[3]

[1] The price of PC decreased at over -17% pa. as Hulten [1990] refers.

[2] See Kuroda, Yoshioka and Shimizu [1987] and Yoshioka [1989]

[3] See for example, Leontief [1970]. And Yoshioka [1989], chap. 5 includes a dynamic model, which is a precedent of this research. Frisch [1933] is pioneering works of the dynamic cyclical model in relations to technical change. The final comments on the innovation describes technical change as a factor in maintaining oscillations.

In the present context, the term "dynamic" means that capital formation processes are endogenously generated in the model. In the dynamic situation, productivity growth or wage increases change commodity prices, not only directly through static spillover effects, but also indirectly through the prices for investment goods which reflect the capital gains or losses.[4]

This research investigates the fluctuation of prices including prices of capital goods through the inter-industry propagation mechanism. The most comprehensive data that are designed to incorporate the inter-industry propagation mechanism is the Leontief's input-output table. Leontief [1970] proposed the dynamic inverse model within the constant input coefficient matrix. I extend the Leontief's dynamic inverse model into the model consistent with the growth accounts.[5]

The early attempts to investigate economic fluctuations through the inter-industry transactions are by Frisch [1933] and [1934]. Frisch [1933] summaries three types of propagation problems: (1) the time lag between production and completing the production of capital goods. Aftalion [1913, 1927] investigates this propagation system. (2) Accumulation of erratic shocks. Slutzky [1927] and Yule [1927] derived several stochastic processes that have oscillation. (3) The innovation as an impulse and its reaction. Frisch sites Schumpeter's business cycle model.

Frisch [1934] considers cycles caused by a random impulse, e.g. Norway's lottery using a transaction model between shoe makers and farmers. But the model assumes fixed prices or given prices under the business cycles.

There have been a lot of contributions on multi-sectoral economic dynamics in the last half of the 20th century.[6] I propose a model with observable variables in which "observable" means it is ready to obtain the parameters or the data from statistical surveys. More precisely, I construct this on the bases of the growth accounts equality, namely the Divisia index, which is now commonly used to calculate the total factor productivity. The

[4]Jorgenson[1963] develops the user cost of capital in the form of maximization of net worth. Jorgenson and Siebert[1968] examines the effects of capital gain on capital accumulation. They find that incorporating capital gains on assets is significant to explain optimal capital accumulation.

[5]The growth accounts are one of the most common indices on prices and quantity, because the Divisia index satisfies with the conditions of Fisher's tests for the ideal index numbers, Richter's invariance axiom, and Hulten's path independence.

[6]For example, Leontief [1970]. I will show the difference from our recent model, described by using the input-output tables. Goodwin [1953] proposed the linear multi-sectoral model, and recently Antonello [1999] described Goodwin's model in terms of an optimal control problem. Nishimura [2001] surveys non-linear dynamics using an optimal growth model with multi-sectoral capital goods.

most difficult data that I assumed is capital goods and the depreciation. I would like to avoid all the difficulty concerned with the measurement of capital and depreciation, thus I adopt the simplest formulation though it is not enough to link the actual statistics with the model.[7] I assumed that capital is a bundle of commodities that can be used over the period, and that the same commodity such as glass, or computer can be utilized as capital goods, intermediate inputs and final demand.

This research attempts to analyze price movements in a general equilibrium framework incorporating the effects of capital losses induced by "Total Factor Productivity"(TFP) growth, and also shows the effects of factor prices, wages and rates of return.

Chapter 1 describes the outline of the model and the definitions of variables. Firstly, I will formulate user cost of capital in the inter-temporal optimization scheme. This is a very simple formulation but it is convenient to extend the model into the full general equilibrium model with households. Basically our analysis uses an open model, which treats wages and interest rates as exogenously given. Next, I am introducing the growth accounting balances based on the use cost of capital previously derived. And the notation has to be extended to establish the model for the whole economy. Finally, I will compare the result with the Leontief's dynamic inverse.

Chapter 2 explores the non-stochastic model. I will extend the model step by step into the general n-sector model. The first case is an one-sector model. It shows continuous price declines if TFP growth is greater than the nominal wage growth. This implies that a rule of nominal wage determination that is fixed to productivity growth may reduce output prices to zero, even if real wages increase. Even in the very simplest situation, the model becomes highly complicated because of the nonlinearity of the differential equations (price equations in the form of growth accounts).

The second case is a two-sector model, in which one of the commodities is not a capital good. In this model, basically the same conclusion can be obtained as that obtained for the one-sector model. Also a simple two-sector model can easily generate unstable variations of price changes, as I will show by numerical experiment.

Thirdly, a general two-sector model is treated. In this general two-sector model, prices change in various patterns, typically in drifting and cyclical motions. This inertial pattern of price variations is due to the nonlinearity of the 2nd order differential equations that describe the model. And relatively high TFP growth or relatively low increase of wages in one sector turns out

[7] Hulten [1990] surveys problems associated with the measurement of capital, and Diewert [1980] surveys aggregation problems on the capital.

to result in the capital losses of investment goods in the both sectors.

In the one-sector model, the economic restrictions on the parameters of the 2nd order differential equation eliminate the possibility of a cyclical pattern of price movements. But in the two-sector model, the inter-industry effects of capital gains or losses often dominate the effects of own sector productivity and wage growth. Thus, cyclical variations of price changes occur. The above results have been attained under the additional restrictive assumptions of constant cost shares and steady growth of both TFP and wages. In the static model, these assumptions result in steady constant growth, and could not cause structural price changes. The present complicated results in this multi-sector model arise from the dynamic effects of investment goods' prices on the output prices.

Finally, I will fully investigate the autonomous two-sector model. The system is completely classified by its determinants and the characteristic roots. There are eight cases that I will show its phase diagrams.

Chapter 3 introduces the stochastic factor into the system. I will describe The brief survey on random process and solution method for the partial differential equations. As van Kampen explains, there are many inappropriate uses of the stochastic differential equations. So I try to review critically the stochastic economic models.

There have been known only a few solutions for the Fokker-Planck equation, which is equivalent to the stochastic differential equation. Therefore I will explain extensively the solution method of the partial differential equations (PDE). I depend on the one-parameter Lie group method to solve the PDE. But the procedure is extremely computationally intensive.

Chapter 4 treats the stochastic model. I will introduce the Langevin effect into the dynamic price equation through the technical change. The equation becomes complicated and will have no strict solution even if we treat one-sector model. But the property of the Langevin equation is examined and the formulation is extended into the n-sector model. The result shows that the impulse effect does not separate from the propagation effect in the stochastic model as Frisch described in 1933.

Finally, in Chapter 5 I will summarize the result and briefly explain the remained issues. There are many problems to be solved.

Chapter 1

Definitions and the model

1.1 User cost from the optimisation

In the introduction I pointed out that a rapid growth of the total factor productivity in a sector may give rise to the increase of the unit cost for other sector through capital loss of the installed equipment, and this may cause volatile price fluctuation in the economy as a whole. In this section, I present a basic equation to show how the prices transmit across the sectors. It is not necessary to provide an optimal control model in order to describe the price interdependency between sectors, but some justification of the definition on cost of capital that I use may be necessary. First I show one of the possible explanations of the cost of capital, and next, introduce the price equations based on the growth accounts of cost.

Consider a cost minimising production sector using materials, labour inputs and capital goods. The producer of a commodity x will minimise the cost C over the planning period $(0, T)$ with a discount rate r.

$$C = \int_0^T c(t)e^{-rt}dt, \tag{1.1}$$

where $c(t)$ is an instant cost and defined as the sum of labour, material and investment costs:

$$c(t) = \boldsymbol{w}'\boldsymbol{l} + \boldsymbol{p_X}'\boldsymbol{x} + \boldsymbol{p_I}'\boldsymbol{x_I}, \tag{1.2}$$

where $\boldsymbol{w}$, $\boldsymbol{p_X}$, $\boldsymbol{p_I}$ denotes n_L wages, n material prices, and n prices of investment goods respectively. $\boldsymbol{l}$, $\boldsymbol{x}$, $\boldsymbol{x_I}$ denotes n_L types of labour, n kinds of materials, and n kinds of investment goods respectively. There are n

kinds of commodity in the economy, where each commodity can be used as either material, investment or consumption.

The producer is subject to the current production technology, which is described as :

$$f(y, l, x, k) = 0, \tag{1.3}$$

where y denotes an output commodity (or commodity vector) that this sector produces, and k denotes capital commodity vector that forms into the sector's equipment and the other durables.

The producer minimises eq (1.1) subject to eq (1.3) and a capital formation equation (1.4)

$$\dot{k} = x_I - \boldsymbol{\delta} k, \tag{1.4}$$

where $\dot{k}$ denotes the time derivative of capital goods vector dk/dt, and $\boldsymbol{\delta}$ denotes a diagonal matrix of depreciation for each capital goods.[1]

$$\boldsymbol{\delta} = \begin{pmatrix} \delta_1 & 0 & \cdots & 0 \\ 0 & \delta_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_n \end{pmatrix}$$

The first order (necessary) conditions are derived from either the calculus of variations or the dynamic programming, expressed as follows:

$$\frac{\delta \mathcal{L}}{\delta l} = e^{-rt}\left(w - \lambda \frac{\partial f}{\partial l}\right) = 0 \tag{1.5}$$

$$\frac{\delta \mathcal{L}}{\delta x} = e^{-rt}\left(p_X - \lambda \frac{\partial f}{\partial x}\right) = 0 \tag{1.6}$$

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta k} &= e^{-rt}\left(\boldsymbol{\delta} p_I - \lambda \frac{\partial f}{\partial k}\right) - \frac{d}{dt}\left(\frac{\partial e^{-rt} c(t)}{\partial \dot{k}}\right) \\ &= e^{-rt}\left(\boldsymbol{\delta} p_I - \lambda \frac{\partial f}{\partial k}\right) + e^{-rt}\left(r p_I - \frac{d p_I}{dt}\right) \\ &= e^{-rt}\left((r\mathbf{I} + \boldsymbol{\delta} - \boldsymbol{\pi}) p_I - \lambda \frac{\partial f}{\partial k}\right) = 0, \end{aligned} \tag{1.7}$$

where λ denotes the Lagrange multiplier, $\mathbf{I}$ denotes the identical matrix of $n \times n$ dimension, $\boldsymbol{\pi}$ denotes the diagonal matrix of $n \times n$ dimension, the

[1] See Hulten and Wykoff [1981] and Hulten [1990] for problems of the formulation. Several other types of depreciation and aggregation formula have been proposed, but many economic theorists do not consider these problems seriously, see for example Nishimura [2001].

diagonal element is capital gain from each capital goods $\dot{p}_{Ii}/p_{Ii}$.

$$\boldsymbol{\pi} = \begin{pmatrix} \frac{d \ln p_{I1}}{dt} & 0 & \cdots & 0 \\ 0 & \frac{d \ln p_{I2}}{dt} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{d \ln p_{In}}{dt} \end{pmatrix}$$

Equations (1.5)–(1.7) show that this first order condition is interpreted as static cost minimisation with $\mathbf{l}$, $\mathbf{x}$, $\mathbf{k}$, if the cost of capital is defined by $(r\mathbf{I} + \boldsymbol{\delta} - \boldsymbol{\pi})\mathbf{p_I}$.

The total cost can be defined with the user cost of capital as follows:

$$C = \mathbf{w}'\mathbf{l} + \mathbf{p_X}'\mathbf{x} + \{(r\mathbf{I} + \boldsymbol{\delta} - \boldsymbol{\pi})\mathbf{p_I}\}'x_I. \tag{1.8}$$

Assume the producer minimises the cost (1.8) with respect to $\mathbf{l}$, $\mathbf{x}$, and $\mathbf{k}$ subject to the production technology (1.3) under the given input prices $\mathbf{w}$, $\mathbf{p_X}$, r, $\boldsymbol{\delta}$ and $\mathbf{p_I}$. The same system of equations as eqs. (1.5)–(1.7) can be obtained from the first order condition.

The user cost of capital is not necessarily derived from the dynamic programming or from the calculus of variations, because the user cost is actually an opportunity cost of capital goods for a unit of period. The opportunity cost consists of the interest earnings from the same amount of investment for securities rp_Ik, the physical depreciation of the capital $\delta p_I k$, minus the capital gain from selling the capital k at the end of a period $\dot{p}_I k$, assuming the existence of a complete capital market for simplicity.

There is a lot of criticism about the formulation of optimisation, and the production functions incorporating capital stock. As Leontief [1982] criticised the transcendental logarithmic production function, its parameters still have “not be[en] identified with those directly observable in the real world”. In this chapter, I would not say that the production technology (1.3) has directly observable parameters. But I would like to mention the equivalence of the total cost whether it is derived from an optimisation with a production function or from a definition of opportunity cost user cost of capital.

1.2 The growth accounting

A similar procedure can be applied to the total cost (1.8), but in this case I take the total derivatives of logarithms of the total cost to decompose per cent change of the total cost into per cent change of the inputs, the input prices and the total productivity change.

First take total derivative of logarithm of the total cost C:

$$\begin{aligned} \mathrm{d}\ln C &= \mathbf{s_L}'\left(\mathrm{d}\ln \boldsymbol{w} + \mathrm{d}\ln \mathbf{l}\right) + \mathbf{s_X}'\left(\mathrm{d}\ln \mathbf{p_X} + \mathrm{d}\ln \mathbf{x}\right) \\ &\quad + \mathbf{s_K}'\left(\mathrm{d}\ln \mathbf{p_K} + \mathrm{d}\ln \mathbf{k}\right), \end{aligned} \tag{1.9}$$

where $\mathbf{s_L}$ denotes the labour's cost share vector of its element $w_i l_i/C$, $\mathbf{s_X}$ denotes the material's cost share vector of its element $p_{Xi}x_i/C$, $\mathbf{s_K}$ denotes the capital's cost share vector of its element $p_{Ki}k_i/C$, and $\mathbf{p_K}$ denotes the user cost of capital $(r\mathbf{I} + \boldsymbol{\delta} - \boldsymbol{\pi})\mathbf{p_I}$.

Next introduce the total factor productivity (TFP) that is the total output per unit of the total inputs, and its growth rate is defined as follows:

$$\mathrm{d}\ln \mathrm{TFP} = \mathrm{d}\ln X - \left(\mathbf{s_L}'\mathrm{d}\ln \mathbf{l} + \mathbf{s_X}'\mathrm{d}\ln \mathbf{x} + \mathbf{s_K}'\mathrm{d}\ln \mathbf{k}\right), \tag{1.10}$$

where $\mathrm{d}\ln X$ denotes the total output index that is usually expressed in differential form as

$$\mathrm{d}\ln X = \mathbf{s_Y}'\mathrm{d}\ln \mathbf{y}.$$

$\mathbf{s_Y}$ denotes the value output share vector of its element $p_i y_i / \sum p_j y_j$, and $\mathbf{y}$ denotes the output vector as before.

Using the definition of the total factor productivity, equation (1.9) becomes as follows:

$$\mathrm{d}\ln C = \mathbf{s_L}'\mathrm{d}\ln \boldsymbol{w} + \mathbf{s_X}'\mathrm{d}\ln \mathbf{p_X} + \mathbf{s_K}'\mathrm{d}\ln \mathbf{p_K} + \mathrm{d}\ln X - \mathrm{d}\ln \mathrm{TFP}$$

$$\mathrm{d}\ln C/X = \mathbf{s_L}'\mathrm{d}\ln \boldsymbol{w} + \mathbf{s_X}'\mathrm{d}\ln \mathbf{p_X} + \mathbf{s_K}'\mathrm{d}\ln \mathbf{p_K} - \mathrm{d}\ln \mathrm{TFP}. \tag{1.11}$$

This simply implies that the growth rate of the unit cost per unit of output is equal to the sum of the growth rate of the wages, the material prices and the user cost of capital, minus the growth rate of the total factor productivity.

The growth rate of the user cost of capital is expressed as the sum of growth rate of its components. Since $r\mathbf{I}$, $\boldsymbol{\delta}$, and $\boldsymbol{\pi}$ are all diagonal matrices, $\mathrm{d}\ln \mathbf{p_K}$ can be expressed using :

$$
\begin{aligned}
d\ln \mathbf{p_K} &= d\ln(r\mathbf{I}+\boldsymbol{\delta}-\boldsymbol{\pi})\mathbf{p_I} \\
&= d\ln\begin{pmatrix}(r+\delta_1-d\ln p_{I1}/dt)p_{I1}\\(r+\delta_2-d\ln p_{I2}/dt)p_{I2}\\\vdots\\(r+\delta_n-d\ln p_{In}/dt)p_{In}\end{pmatrix} \\
&= \begin{pmatrix}d\ln(r+\delta_1-d\ln p_{I1}/dt)+d\ln p_{I1}\\d\ln(r+\delta_2-d\ln p_{I2}/dt)+d\ln p_{I2}\\\vdots\\d\ln(r+\delta_n-d\ln p_{In}/dt)+d\ln p_{In}\end{pmatrix} \\
&= \begin{pmatrix}\frac{d\ln r+d\ln\delta_1-d^2\ln p_{I1}/dt}{r+\delta_1-d\ln p_{I1}/dt}+d\ln p_{I1}\\\frac{d\ln r+d\ln\delta_2-d^2\ln p_{I2}/dt}{r+\delta_2-d\ln p_{I2}/dt}+d\ln p_{I2}\\\vdots\\\frac{d\ln r+d\ln\delta_n-d^2\ln p_{In}/dt}{r+\delta_n-d\ln p_{In}/dt}+d\ln p_{In}\end{pmatrix}
\end{aligned}
$$

$$
d\ln \mathbf{p_K} = (r\mathbf{I}+\boldsymbol{\delta}-\boldsymbol{\pi})^{-1}\left((dr\mathbf{I}+d\boldsymbol{\delta})\mathbf{1}-d^2\ln\mathbf{p_I}/dt\right)+d\ln\mathbf{p_I}, \tag{1.12}
$$

where $\mathbf{1}$ denotes the column vector of one.

Before inserting (1.12) into (1.11), remember that the above formulation is for a sector, which is to be considered as the j-th sector.

I will introduce the following assumptions for simplicity.

Assumption: No joint production The j-th sector produces the single output X_j for all j. This means that X_j=y in the previous notation for y.

Assumption: Average cost Average cost of production C_j/X_j for the j-th sector is equal to the output price of that sector p_j.

$$
\frac{C_j}{X_j} = p_j, \quad (j = 1,\ldots,n).
$$

Assumption: Notation of the commodity Each output can be used as a material and also as a capital goods, there is n kinds of commodities in the economy. Prices for investment goods $\mathbf{p_I}$, for materials $\mathbf{p_X}$, and for outputs $\mathbf{p}$ are no longer to be distinguished.

$$
\mathbf{p} = \mathbf{p_I} = \mathbf{p_X}
$$

Using these assumptions, the balance between input cost and output price is written as follows:

$$\begin{aligned} d\ln p_j \quad = \quad & \mathbf{s_L}_j' d\ln \boldsymbol{w} + \mathbf{s_X}_j' d\ln \mathbf{p} \\ & + \mathbf{s_K}_j' \left(r_j \mathbf{I} + \boldsymbol{\delta}_j - \boldsymbol{\pi}\right)^{-1} \left((dr_j \mathbf{I} + d\boldsymbol{\delta}_j)\mathbf{1} - d^2 \ln \mathbf{p}/dt\right) \\ & + \mathbf{s_K}_j' d\ln \mathbf{p} - d\ln TFP_j, \quad (j = 1, \ldots, n). \end{aligned} \tag{1.13}$$

Divide both sides of the equation by dt, then the equation is expressed in terms of the growth rate per unit of time.

$$\begin{aligned} \frac{d\ln p_j}{dt} \quad = \quad & \mathbf{s_L}_j' \frac{d\ln \boldsymbol{w}}{dt} + \mathbf{s_X}_j' \frac{d\ln \mathbf{p}}{dt} \\ & + \mathbf{s_K}_j' \left(r_j \mathbf{I} + \boldsymbol{\delta}_j - \boldsymbol{\pi}\right)^{-1} \left(\left(\frac{dr_j}{dt}\mathbf{I} + \frac{d\boldsymbol{\delta}_j}{dt}\right)\mathbf{1} - \frac{d^2 \ln \mathbf{p}}{dt^2}\right) \\ & + \mathbf{s_K}_j' \frac{d\ln \mathbf{p}}{dt} - \frac{d\ln TFP_j}{dt}, \quad (j = 1, \ldots, n). \end{aligned} \tag{1.14}$$

Now we can introduce the balance equation of the whole economy.

1.3 Analytical framework

In this section, we derive the price equations of the whole economy in the growth rate form. First of all, the notations are summerised as follows.

x_{ij} : input of the i-th good to produce the j-th good.

p_i : price of the i-th good.

l_j : labor input in the j–th sector.

w_j : wage rate of l_j

k_{ij} : stock of the i-th investment good used to produce the j-th good.

$p_{K_{ij}}$: user cost of k_{ij} .

r_j : interest rate of the j-th sector which produces the j-th good.

δ_{ij} : depreciation rate of k_{ij}.

C_j : total factor cost of the j-th sector.

X_j : aggregated products of the j-th sector. $X_j \stackrel{\text{def}}{=} \sum_{i=1}^{n} x_{ij}$.

TFP_j : total factor productivity of the j-th sector, defined by total input price changes minus total output price changes, which is an equivalent definition to total output growth minus total input growth under the stable profit rate.

S_{Xij} : cost share of the i-th input in the j-th sector.

$$S_{Xij} \stackrel{\text{def}}{=} \frac{p_i x_{ij}}{C_j}$$

S_{Kij} : cost share of the i-th investment good in the j-th sector.

$$S_{Kij} \stackrel{\text{def}}{=} \frac{p_{Kij} k_{ij}}{C_j}$$

S_{Lj} : labor's cost share of the j-th sector.

$$S_{Lj} \stackrel{\text{def}}{=} \frac{w_j l_j}{C_j}$$

Total cost of the j-th sector is as follows:

$$C_j \stackrel{\text{def}}{=} \sum_{i=1}^{n} p_i x_{ij} + w_j l_j + \sum_{i=1}^{n} p_{Kij} k_{ij}, \tag{1.15}$$

where $p_{Kij} \stackrel{\text{def}}{=} p_i \left(r_j + \delta_{ij} - d\ln p_i/dt\right)$.

In the growth rate form, total cost of the j-th sector is written as in the next equation:

$$\frac{d\ln C_j/X_j}{dt} \stackrel{\text{def}}{=} \sum_{i=1}^{n} S_{Xij} \frac{d\ln p_i}{dt} + \sum_{i=1}^{n} S_{Kij} \frac{d\ln p_{Kij}}{dt} + S_{Lj} \frac{d\ln w_j}{dt} - \frac{d\ln \mathrm{TFP}_j}{dt}. \tag{1.16}$$

Under perfect competition, we have, $p_j = C_j/X_j$, and hece the price equation becomes as follows:

$$\frac{d\ln p_j}{dt} = \sum_{i=1}^{n} S_{Xij} \frac{d\ln p_i}{dt} + \sum_{i=1}^{n} S_{Kij} \frac{d\ln p_{Kij}}{dt} + S_{Lj} \frac{d\ln w_j}{dt} - \frac{d\ln \mathrm{TFP}_j}{dt}. \tag{1.17}$$

We find the term $d\ln p_{Kij}/dt$ (change in the cost of capital) in (1.17), which needs more calculation to interpret. The change in the cost of capital can be broken down into the changes in interest rates, depreciation, and capital gains or losses. Differenciating $p_{K_{ij}}$ gives:

$$\frac{d\ln p_{K_{ij}}}{dt} = \frac{d\ln p_i}{dt} + \frac{p_i r_j}{p_{K_{ij}}}\frac{d\ln r_j}{dt} + \frac{p_i \delta_{ij}}{p_{K_{ij}}}\frac{d\ln \delta_{ij}}{dt} - \frac{p_i}{p_{K_{ij}}}\frac{d^2\ln p_i}{dt^2}. \quad (1.18)$$

If (1.18) is substituted into (1.17) for each sector, using the definiton of $p_{K_{ij}}$, we get:

$$\begin{aligned} \textstyle\sum_{i=1}^{n} S_{K_{ij}} \frac{1}{r_j+\delta_{ij}-d\ln p_i/dt} \frac{d^2\ln p_i}{dt^2} + \sum_{i=1}^{n} (\Delta_{ij} - S_{X_{ij}} - S_{K_{ij}}) \frac{d\ln p_i}{dt} \\ = \quad S_{L_j}\frac{d\ln w_j}{dt} + \textstyle\sum_{i=1}^{n} S_{K_{ij}}\frac{d\ln p_i}{dt}\left(\frac{dr_j}{dt} + \frac{d\delta_{ij}}{dt}\right) - \frac{d\ln TFP_j}{dt} \\ (j = 1, \ldots, n), \end{aligned}$$

where Δ_{ij} is a Kronecker's delta, that is $\Delta_{ij} = 0$ when $i = j$, and $\Delta_{ii} = 1$ otherwise.

Equation (1.19) are the basic equations of our analysis in this book. $p_{K_{ij}}$ is on the right hand side of (1.19) and hece (1.19) is not a reduced form equation. Furthermore, cost–shares can vary in some or all of the prices. If we assume that cost–shares are constant, it means that the underlying production function is log-linear, i.e. Cobb–Douglas type.

1.4 The system in vector notation

For convenience, I would like to introduce vector notation for the system. In the economy as a whole, the above system of equations is described as follows:

$$\begin{aligned} \frac{d\ln \mathbf{p}}{dt} \quad = \quad & \mathbf{S_X}\frac{d\ln \mathbf{p}}{dt} + \mathbf{S_K}\frac{d\ln \mathbf{p}}{dt} - \mathbf{Sr}\frac{d^2\ln \mathbf{p}}{dt^2} \\ & + \mathbf{S_L}\frac{d\ln \mathbf{w}}{dt} + \left(\frac{\mathbf{R}}{dt}\mathbf{Sr} + \mathbf{Sd}\right)\mathbf{1} - \frac{d\ln \mathbf{TFP}}{dt}. \end{aligned} \quad (1.19)$$

The growth account equations (1.19) corresponds to the Leontief's price equation (1.23), as we see later in this chapter.

Again, definitions of the matrices are as follows:

$$\mathbf{S_L} = \begin{pmatrix} s_{L11} & s_{L21} & \cdots & s_{Ln_L1} \\ s_{L12} & s_{L22} & \cdots & s_{Ln_L2} \\ \vdots & \vdots & & \vdots \\ s_{L1n} & s_{L2n} & \cdots & s_{Ln_Ln} \end{pmatrix}$$

$$\mathbf{S_X} = \begin{pmatrix} s_{X11} & s_{X21} & \cdots & s_{Xn1} \\ s_{X12} & s_{X22} & \cdots & s_{Xn2} \\ \vdots & \vdots & & \vdots \\ s_{X1n} & s_{X2n} & \cdots & s_{Xnn} \end{pmatrix}$$

$$\frac{d\mathbf{R}}{dt} = \begin{pmatrix} \frac{dr_1}{dt} & 0 & \cdots & 0 \\ 0 & \frac{dr_2}{dt} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{dr_n}{dt} \end{pmatrix}$$

$$\mathbf{Sr} = \begin{pmatrix} \frac{s_{K11}}{r_1+\delta_{11}-\frac{d\ln p_1}{dt}} & \frac{s_{K21}}{r_1+\delta_{21}-\frac{d\ln p_2}{dt}} & \cdots & \frac{s_{Kn1}}{r_1+\delta_{n1}-\frac{d\ln p_n}{dt}} \\ \frac{s_{K12}}{r_2+\delta_{12}-\frac{d\ln p_1}{dt}} & \frac{s_{K22}}{r_2+\delta_{22}-\frac{d\ln p_2}{dt}} & \cdots & \frac{s_{Kn2}}{r_2+\delta_{n2}-\frac{d\ln p_n}{dt}} \\ \vdots & \vdots & & \vdots \\ \frac{s_{K1n}}{r_n+\delta_{1n}-\frac{d\ln p_1}{dt}} & \frac{s_{K2n}}{r_n+\delta_{2n}-\frac{d\ln p_2}{dt}} & \cdots & \frac{s_{Knn}}{r_n+\delta_{nn}-\frac{d\ln p_n}{dt}} \end{pmatrix}$$

$$\mathbf{Sd} = \begin{pmatrix} \frac{s_{K11}\frac{d\delta_{11}}{dt}}{r_1+\delta_{11}-\frac{d\ln p_1}{dt}} & \frac{s_{K21}\frac{d\delta_{21}}{dt}}{r_1+\delta_{21}-\frac{d\ln p_2}{dt}} & \cdots & \frac{s_{Kn1}\frac{d\delta_{n1}}{dt}}{r_1+\delta_{n1}-\frac{d\ln p_n}{dt}} \\ \frac{s_{K12}\frac{d\delta_{12}}{dt}}{r_2+\delta_{12}-\frac{d\ln p_1}{dt}} & \frac{s_{K22}\frac{d\delta_{22}}{dt}}{r_2+\delta_{22}-\frac{d\ln p_2}{dt}} & \cdots & \frac{s_{Kn2}\frac{d\delta_{n2}}{dt}}{r_2+\delta_{n2}-\frac{d\ln p_n}{dt}} \\ \vdots & \vdots & & \vdots \\ \frac{s_{K1n}\frac{d\delta_{1n}}{dt}}{r_n+\delta_{1n}-\frac{d\ln p_1}{dt}} & \frac{s_{K2n}\frac{d\delta_{2n}}{dt}}{r_n+\delta_{2n}-\frac{d\ln p_2}{dt}} & \cdots & \frac{s_{Knn}\frac{d\delta_{nn}}{dt}}{r_n+\delta_{nn}-\frac{d\ln p_n}{dt}} \end{pmatrix}$$

$$\mathbf{S_K} = \begin{pmatrix} s_{K11} & s_{K21} & \cdots & s_{Kn1} \\ s_{K12} & s_{K22} & \cdots & s_{Kn2} \\ \vdots & \vdots & & \vdots \\ s_{K1n} & s_{K2n} & \cdots & s_{Knn} \end{pmatrix}.$$

Definitions of the vectors are as follows:

$$\frac{d\ln\mathbf{p}}{dt} = \begin{pmatrix} \frac{d\ln p_1}{dt} \\ \frac{d\ln p_2}{dt} \\ \vdots \\ \frac{d\ln p_n}{dt} \end{pmatrix}$$

$$\frac{d\ln\boldsymbol{w}}{dt} = \begin{pmatrix} \frac{d\ln w_1}{dt} \\ \frac{d\ln w_2}{dt} \\ \vdots \\ \frac{d\ln w_{n_L}}{dt} \end{pmatrix}$$

$$\frac{d\ln \mathbf{TFP}}{dt} = \begin{pmatrix} \frac{d\ln TFP_1}{dt} \\ \frac{d\ln TFP_2}{dt} \\ \vdots \\ \frac{d\ln TFP_n}{dt} \end{pmatrix}$$

$$\frac{d^2\ln \mathbf{p}}{dt^2} = \begin{pmatrix} \frac{d^2\ln p_1}{dt^2} \\ \frac{d^2\ln p_2}{dt^2} \\ \vdots \\ \frac{d^2\ln p_n}{dt^2} \end{pmatrix}.$$

The other definitions of variables are as follows:

$$s_{Lij} = \frac{w_i l_{ij}}{C_j}, \quad s_{Xij} = \frac{p_i x_{ij}}{C_j}$$

$$\begin{aligned} S_{Kij} &= s_{Kij} = \frac{p_{Ki} k_{ij}}{C_j} \\ &= \frac{p_i\left(r_j + \delta_{ij} - \frac{d\ln p_i}{dt}\right) k_{ij}}{C_j}. \end{aligned}$$

s_{Lij} is the i-th labour's cost share in the j-th sector, likewise s_{Xij} is the i-th material's cost share in the j-th sector, and s_{Kij} is the i-th capital goods' cost share in the j-th sector. There are n_L types of labour, n types of materials and capital goods.

After transposing the terms with $d\ln \mathbf{p}/dt$ to the left hand side in (1.19), the system becomes as follows:

$$\begin{aligned} \mathbf{Sr}\frac{d^2\ln \mathbf{p}}{dt^2} + (\mathbf{I} - \mathbf{S_X} - \mathbf{S_K})\frac{d\ln \mathbf{p}}{dt} &= -\frac{d\ln \mathbf{TFP}}{dt} + \mathbf{S_L}\frac{d\ln \boldsymbol{w}}{dt} \\ &\quad + \left(\frac{d\mathbf{R}}{dt}\mathbf{Sr} + \mathbf{Sd}\right)\mathbf{1}. \end{aligned} \tag{1.20}$$

I investigated this system of equations, which is non-linear with respect to $d\ln \mathbf{p}/dt$, because $\mathbf{Sr}$ includes $d\ln \mathbf{p}/dt$, even if all the cost share matrices are assumed to be constant.[2] This system is homogeneous degree zero in prices, because the equations do not change if all the prices grow at $\lambda > 0$, $\lambda\mathbf{p}$, $\lambda\boldsymbol{w}$, when

$$(\mathbf{I} - \mathbf{S_X} - \mathbf{S_K} - \mathbf{S_L})\mathbf{1} = 0.$$

If wages $\boldsymbol{w}$ are adjusted to cancel the effect of the total factor productivity fluctuation, while the interest rate and the depreciation keep unchanged, the system becomes autonomous.

[2]Hayami[1993] proposed the same equations as a different form. Although my previous formulation contains errors of notations, the two sector model in that paper is exactly the same as this. As to nonlinear dynamical systems see for example, Guckenheimer Holms [1990].

1.5 An implication to the Leontief's dyanamic inverse

Leontief dynamics [1970] is the system of equations that describes physical balance of demand and supply.

$$\mathbf{x}_t - \mathbf{A}_t\mathbf{x}_t - \mathbf{B}_{t+1}(\mathbf{x}_{t+1} - \mathbf{x}_t) = \mathbf{c}_t, \tag{1.21}$$

where $\mathbf{A}_t$ denotes the input coefficient matrix at t, and $\mathbf{B}_{t+1}$ denotes the capital coefficient matrix, "capital goods produced in year t are assumed to be installed and put into operation in the next year t+1", and $\mathbf{c}_t$ denotes final demand vector.[3]

Leontief [1970] proposes the following associated cost accounting, which is dual to (1.21).

$$\mathbf{p}_t = \mathbf{A}_t'\mathbf{p}_t + (1 + r_{t-1})\mathbf{B}_t'\mathbf{p}_{t-1} - \mathbf{B}_{t+1}'\mathbf{p}_t + \boldsymbol{\nu}_t, \tag{1.22}$$

where r_{t-1} denotes the annual money rate of interest prevailing in that year, and $\boldsymbol{\nu}_t$ a vector of the value added per unit of its output. This equation can be rewritten as follows:

$$\mathbf{p}_t = \mathbf{A}_t'\mathbf{p}_t + (r_{t-1}\mathbf{I} - \boldsymbol{\pi})\mathbf{B}_t'\mathbf{p}_{t-1} - (\mathbf{B}_{t+1}' - \mathbf{B}_t')\mathbf{p}_t + \boldsymbol{\nu}_t, \tag{1.23}$$

This system of equations implies that the price of output $\mathbf{p}_t$ is equal to the sum of the unit cost of materials $\mathbf{A}_t'\mathbf{p}_t$, the value added per output $\boldsymbol{\nu}_t$, which includes employment cost and depreciation, and the user cost of net capital $(r_{t-1}\mathbf{I}-\boldsymbol{\pi})\mathbf{B}_t'\mathbf{p}_{t-1}$, minus the technical change $(\mathbf{B}_{t+1}'-\mathbf{B}_t')\mathbf{p}_t$. The dynamic inverse price equation is precisely analogous to our price equation (1.8) except that it ignores the depreciation. If we take further time diffrence of (1.23), we can obtain the descrete approximaiton of our system.

[3] Leontief [1970], and reprinted in Leontief [1986] p.295.

Chapter 2

Non-stochastic models

In this chapter, I would like to explore four examples of non-stochasitic models. First of all, the simple one sector model is to be analysed. Next, I derive two sector model that has the same property as the one sector model. And third, I calculate the general two sector model with various parameter values. The simulations will reveal the basic properties of this model. Finally, I investigate the autonomouse system of this model, which presents the system's property on the price propagation.

2.1 One-sector model

The price equation system (1.19) takes the simplest form in the one-sector model.

$$\begin{aligned} & S_K \frac{p}{\rho} \frac{d^2 \ln p}{dt^2} + (1 - S_X - S_K) \frac{d \ln p}{dt} \\ = \; & S_L \frac{d \ln w}{dt} + S_K \frac{1}{p_K} \left(\frac{dr}{dt} + \frac{d\delta}{dt} \right) - \frac{d \ln \mathrm{TFP}}{dt} \end{aligned} \tag{2.1}$$

Next substituting $p_K = p(r + \delta - d \ln p / dt)$ into the above equation, we obtain

$$\begin{aligned}
&\frac{d^2 \ln p}{dt^2} - \frac{(1 - S_X - S_K)}{S_K}\left(\frac{d \ln p}{dt}\right)^2 \\
+ &\left(\frac{1}{S_K}\left(S_L \frac{d \ln w}{dt} - \frac{d \ln TFP}{dt}\right) + (r + \delta)\,\frac{1 - (S_X + S_K)}{S_K}\right)\frac{d \ln p}{dt} \\
= &\left(\frac{dr}{dt} + \frac{d\delta}{dt}\right) + \frac{(r + \delta)}{S_K}\left(S_L \frac{d \ln w}{dt} - \frac{d \ln TFP}{dt}\right). \qquad (2.2)
\end{aligned}$$

For analytical convenience, we introduce the following parameters and variables.

$$a = (1 - S_X - S_K)/S_K > 0$$

$$R(t) = r + \delta > 0$$

$$R'(t) = dr/dt + d\delta/dt$$

$$s(t) = [S_L \, d \ln w/dt - d \ln TFP/dt]\,/S_K$$

$$y(t) = d \ln p/dt$$

Then equation (2.2) becomes the following first order ordinary differential equation:

$$y'(t) - a y(t)^2 + [s(t) + aR(t)]\, y(t) = R'(t) + R(t)s(t)$$

Equation (2.3) is a nonlinear differential equation of the Riccati type. Thus it is not possible to solve it generally by the integration method (see Yoshida [1977]).

In this section, we assume that $R(t)$ and $s(t)$ are constant over time. We transform variable $y(t)$ as follows: $-a \cdot y(t) = u'(t)/u(t)$, where $u'(t) = du(t)/dt$. Then equation (2.3) becomes the following 2nd order ordinary linear differential equation:

$$u''(t) + [s + a \cdot R]\, u'(t) + a \cdot R \cdot s \cdot u(t) = 0. \qquad (2.3)$$

Equation (2.3) has two characteristic roots of real numbers, because its determinant D is positive.

$$D = (s - aR)^2 \geq 0 \qquad (2.4)$$

And the solution of (2.3) is obtained by (2.5),

$$\begin{aligned} u(t) &= \frac{1}{s-a\cdot R}\left([s\cdot u(0)+u'(0)]\exp\left(-a\cdot R\cdot t\right)\right. \\ &\quad \left. -\left[a\cdot R\cdot u(0)+u'(0)\right]\exp\left(-s\cdot t\right)\right) \end{aligned} \tag{2.5}$$

where $u(0)$ and $u'(0)$ are the initial conditions and $p(t) = C^{1/a}/u(t)^{1/a}$. Because $a \cdot R > 0$, the first term of the left hand side of (2.5) converges to 0, as time t goes to infinity. Whether the second term of the left hand side of (2.5) converges or diverges depends on whether s is positive or negative. Thhe initial condition $[a \cdot R \cdot u(0) + u'(0)]$ is positive in sign, because it is reduced to $r+\delta-d\ln p(0)/dt$, which is positive in normal situations. If s is negative, then u(t) becomes infinitely large, and p(t)($= p(0)[u(0)/u(t)]^{1/a}$) converges to 0. On the other hand, if s is positive, u(t) and p(t) both converge.

2.2 Simplified two-sector model

In a two sector model, the equation (1.19) can be written explicitly as follows:

$$\begin{aligned} & S_{K11}\left(r_1+\delta_{21}-\frac{d\ln p_2}{dt}\right)\frac{d^2\ln p_1}{dt^2}+S_{K21}\left(r_1+\delta_{11}-\frac{d\ln p_1}{dt}\right)\frac{d^2\ln p_2}{dt^2} \\ + & \left(r_1+\delta_{11}-\frac{d\ln p_1}{dt}\right)\left(r_1+\delta_{21}-\frac{d\ln p_2}{dt}\right) \\ \times & \left((1-(S_{X11}+S_{K11}))\frac{d\ln p_1}{dt}-(S_{X21}+S_{K21})\frac{d\ln p_2}{dt}\right) \\ = & S_{K11}\left(r_1+\delta_{21}-\frac{d\ln p_2}{dt}\right)\left(\frac{dr_1}{dt}+\frac{d\delta_{11}}{dt}\right) \\ + & S_{K21}\left(r_1+\delta_{11}-\frac{d\ln p_1}{dt}\right)\left(\frac{dr_2}{dt}+\frac{d\delta_{21}}{dt}\right) \\ + & \left(r_1+\delta_{11}-\frac{d\ln p_1}{dt}\right)\left(r_1+\delta_{21}-\frac{d\ln p_2}{dt}\right)\left(S_{L1}\frac{d\ln w_1}{dt}-\frac{d\ln TFP_1}{dt}\right) \end{aligned} \tag{2.6}$$

$$\begin{aligned} & S_{K12}\left(r_2+\delta_{22}-\frac{d\ln p_1}{dt}\right)\frac{d^2\ln p_1}{dt^2}+S_{K22}\left(r_2+\delta_{12}-\frac{d\ln p_2}{dt}\right)\frac{d^2\ln p_2}{dt^2} \\ + & \left(r_2+\delta_{12}-\frac{d\ln p_1}{dt}\right)\left(r_2+\delta_{22}-\frac{d\ln p_2}{dt}\right) \\ \times & \left((1-(S_{X22}+S_{K22}))\frac{d\ln p_2}{dt}-(S_{X12}+S_{K12})\frac{d\ln p_1}{dt}\right) \\ = & S_{K12}\left(r_2+\delta_{22}-\frac{d\ln p_2}{dt}\right)\left(\frac{dr_1}{dt}+\frac{d\delta_{11}}{dt}\right) \\ + & S_{K22}\left(r_2+\delta_{12}-\frac{d\ln p_1}{dt}\right)\left(\frac{dr_2}{dt}+\frac{d\delta_{22}}{dt}\right) \\ + & \left(r_2+\delta_{12}-\frac{d\ln p_1}{dt}\right)\left(r_2+\delta_{22}-\frac{d\ln p_2}{dt}\right)\left(S_{L2}\frac{d\ln w_2}{dt}-\frac{d\ln TFP_2}{dt}\right) \end{aligned} \tag{2.7}$$

For simplification, it is assumed that the 2nd sector does not produce investment goods, thus we set $S_{K21}, S_{K22} = 0$. Equations (2.6) and (2.7) become as follows:

$$\begin{aligned} & S_{K11}\tfrac{d^2 \ln p_1}{dt^2} + \left(r_1 + \delta_{11} - \tfrac{d \ln p_1}{dt}\right)\left(1 - (S_{X11} + S_{K11})\right)\tfrac{d \ln p_1}{dt} \\ & = S_{K11}\left(\tfrac{dr_1}{dt} + \tfrac{d\delta_{11}}{dt}\right) + \left(r_1 + \delta_{11} - \tfrac{d \ln p_1}{dt}\right)\left(S_{L1}\tfrac{d \ln w_1}{dt} - \tfrac{d \ln TFP_1}{dt}\right) \end{aligned} \tag{2.8}$$

$$\begin{aligned} & S_{K12}\tfrac{d^2 \ln p_1}{dt^2} \\ - & \left(r_2 + \delta_{12} - \tfrac{d \ln p_1}{dt}\right)\left((S_{X12} + S_{K12})\tfrac{d \ln p_1}{dt} - (1 - S_{X22})\tfrac{d \ln p_2}{dt}\right) \\ = & \; S_{K12}\left(\tfrac{dr_2}{dt} + \tfrac{d\delta_{12}}{dt}\right) + \left(r_2 + \delta_{12} - \tfrac{d \ln p_1}{dt}\right)\left(S_{L2}\tfrac{d \ln w_2}{dt} - \tfrac{d \ln TFP_2}{dt}\right) \end{aligned} \tag{2.9}$$

Solving (2.8) for $d^2 \ln p_1 / dt^2$, and substituting this expersson into (2.9), we obtain

$$\begin{aligned} & S_{K11}\left(1 - W_{a22}\right)\left(r_2 + \delta_{12} - \frac{d \ln p_1}{dt}\right)\frac{d \ln p_2}{dt} \\ = \; & \left(S_{K11}S_{X12} + S_{K12}\left(1 - S_{X11}\right)\right)\left(\frac{d \ln p_1}{dt}\right)^2 \\ + \; & \Bigg[S_{K11}\left(S_{X12} + S_{K12}\right)\left(r_2 + \delta_{12}\right) \\ & + S_{K12}\left(r_1 + \delta_{11}\right)\left(1 - (S_{X11} + S_{K11})\right) \\ & - S_{K11}\left(S_{L2}\frac{d \ln w_2}{dt} - \frac{d \ln TFP_2}{dt}\right) \\ & + S_{K12}\left(S_{L1}\frac{d \ln w_1}{dt} - \frac{d \ln TFP_1}{dt}\right)\Bigg]\frac{d \ln p_1}{dt} \\ + \; & S_{K11}S_{K12}\left(\frac{dr_2}{dt} + \frac{\delta_{12}}{dt} - \frac{dr_1}{dt} - \frac{dr_1}{dt} - \frac{d\delta_{11}}{dt}\right) \\ - \; & S_{K12}\left(r_1 + \delta_{11}\right)\left(S_{L1}\frac{d \ln w_1}{dt} - \frac{d \ln TFP_1}{dt}\right) \\ + \; & S_{K11}\left(r_2 + \delta_{12}\right)\left(S_{L2}\frac{d \ln w_2}{dt} - \frac{d \ln TFP_2}{dt}\right). \end{aligned} \tag{2.10}$$

If we assume further the following restrictions, (2.10) will take a simpler form as in the following (2.11).

$$S_X = S_{X12} = S_{X11},$$

$$S_K = S_{K11} = S_{K12},$$

$$R = r_2 + \delta_{12} = r_1 + \delta_{11},$$

$$\frac{d\ln p_2}{dt} = \frac{1}{1-S_X}\left(\frac{d\ln p_1}{dt} - S_{L1}\frac{d\ln w_1}{dt} + S_{L2}\frac{d\ln w_2}{dt} + \frac{d\ln TFP_1}{dt} - \frac{d\ln TFP_2}{dt}\right) \quad (2.11)$$

As shown in (2.8), the equation for the 1–st sector is the same equation as the one derived for the one-sector model in section 2.1. Thus the price movements of the 1–st sector are described by (2.8). The price movements of the 2–nd sector are related to the 1–st sector price variations, as shown in (2.11). Equation (2.11) shows that the price movements of the 2–nd sector differ from those of the 1–st sector in terms of the gaps of wage changes and TFP changes. If wage and TFP change in the same direction, the relative prices of the two sectors change proportionally. Then the ultimate price levels of both sectors are determined by the difference in the wage growth and TFP in the 1–st sector.

2.3 Two-sector model in general

In this section, we explore the two-sector model in it's general form. Equations (2.6) and (2.7) describe the system of a two-sector model in general. To solve this system iteratively, the system is first transformed into a 4 dimensional 1–st order differential equation. Second, we use as the method of solving the differential equation the Adams-Bashforth and Adams–Moulton method.[1]

Afeter applying this procedure, (2.6) and (2.7) become as follows:

[1] I use the Adams–Bashforth methods as predictor, and the Adams–Moulton methods as corrector. This predictor–corrector integration method for solving differential equations is rather complicated. Therefore some authors like Press et. al. [1988] criticize its efficiency. But because of its accuracy and its stability to apply the stiff equation which has a large difference between minimum and maximum chracteristic roots, other authors like Kubíček and Marek[1983] or Parker and Chua[1989] recommend this method. All the programs in this research are coded in C.

$$\begin{pmatrix} \frac{d^2 \ln p_1}{dt^2} \\ \frac{d^2 \ln p_2}{dt^2} \end{pmatrix} = \frac{1}{D} \begin{pmatrix} S_{K22}\left(R_{12} - \frac{d\ln p_1}{dt}\right) - S_{K21}\left(R_{11} - \frac{d\ln p_1}{dt}\right) \\ -S_{K12}\left(R_{22} - \frac{d\ln p_2}{dt}\right) S_{K11}\left(R_{21} - \frac{d\ln p_2}{dt}\right) \end{pmatrix}$$
$$\times \begin{pmatrix} \left(R_{11} - \frac{d\ln p_1}{dt}\right)\left(R_{21} - \frac{d\ln p_2}{dt}\right) \\ \times \left[s_1 - (1 - S_{X11} - S_{K11})\frac{d\ln p_1}{dt} + (S_{X21} + S_{K21})\frac{d\ln p_2}{dt}\right] \\ \left(R_{12} - \frac{d\ln p_1}{dt}\right)\left(R_{22} - \frac{d\ln p_2}{dt}\right) \\ \times \left[s_2 - (1 - S_{X22} - S_{K22})\frac{d\ln p_2}{dt} + (S_{X12} + S_{K12})\frac{d\ln p_1}{dt}\right] \end{pmatrix}$$
$$+ \begin{pmatrix} \frac{dr_1}{dt} \\ \frac{dr_2}{dt} \end{pmatrix} + \begin{pmatrix} S_{K11}\left(R_{21} - \frac{d\ln p_2}{dt}\right)\frac{d\delta_{11}}{dt} \\ +S_{K21}\left(R_{11} - \frac{d\ln p_1}{dt}\right)\frac{d\delta_{21}}{dt} \\ S_{K12}\left(R_{22} - \frac{d\ln p_2}{dt}\right)\frac{d\delta_{12}}{dt} \\ +S_{K22}\left(R_{12} - \frac{d\ln p_1}{dt}\right)\frac{d\delta_{22}}{dt} \end{pmatrix}. \tag{2.12}$$

where,

$$D \stackrel{\text{def}}{=} S_{K11}S_{K22}\left(R_{12} - \frac{d\ln p_1}{dt}\right)\left(R_{22} - \frac{d\ln p_2}{dt}\right) - S_{K12}S_{K21}\left(R_{11} - \frac{d\ln p_1}{dt}\right)\left(R_{22} - \frac{d\ln p_2}{dt}\right). \tag{2.13}$$

$$R_{ij} \stackrel{\text{def}}{=} r_j + \delta_{ij}. \tag{2.14}$$
$$s_1 \stackrel{\text{def}}{=} S_{L1}\frac{d\ln w_1}{dt} - \frac{d\ln TFP_1}{dt}. \tag{2.15}$$
$$s_2 \stackrel{\text{def}}{=} S_{L2}\frac{d\ln w_2}{dt} - \frac{d\ln TFP_2}{dt}. \tag{2.16}$$

As previously defined, D is a function of $d\ln p_1/dt$ and $d\ln p_2/dt$. Therefore, we can get the solutions only for non-singular D, and there is an unresolved issue of whether D is always non-singular or not. Some examples for illustrative purposees are found in Figure 2.1–2.6. Figugre 2.6 shows initially the irregular movements of price variations. It seems that values of the s_1 and s_2 play an important role in determining the feature of output price changes.

In these numerical experiments, the parameters of S_{Xij}, S_{Kij}, S_{L1}, S_{L2} are aggregated and roughly calculated from *Input–Output Table 1985*(especially capital formation matrix, Japan Ministry of Internatal Trade and

Industry). Sector 1 is assumed to produce manufacturing commodities, and sector 2 is assumed to produce service and construction activities. Both sectors depend on capital goods produced in both sectors. For our purpose to illustrate the heuristic mechanism of cyclical price variations, these tentatively chosen values suffice but further data elaboration might be required to display realistic price fluctuations of the economic system. However, s_1, s_2 provide a reasonable range of values for simulations.

Figure 2.1 shows very normal pattern of price variations. But the interactive effects between the two industries shows the downward pressure of price changes.

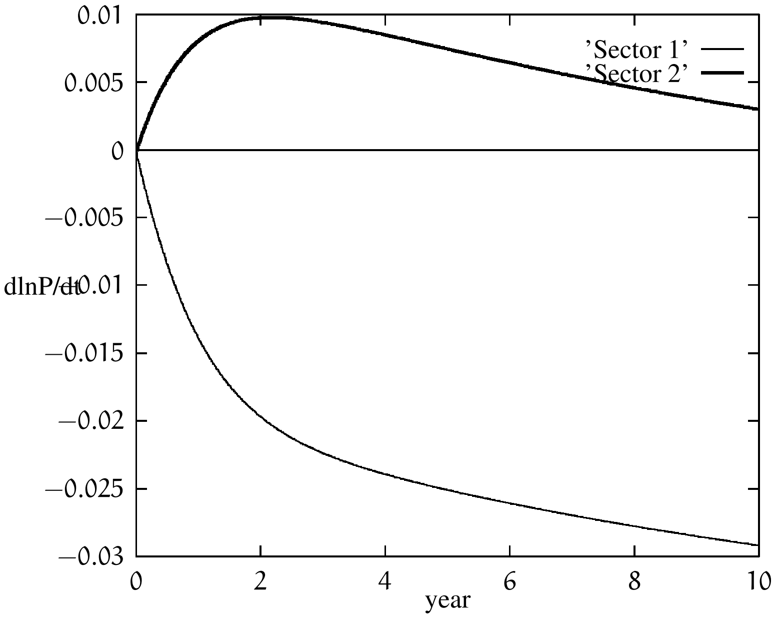
Figure 2.2 gives an example of a capital gain effect of the 2nd industry because of the high inflation rate of the 1st commodity. Its effect on decreasing costs lasts for about four years in this simulation.

Figure 2.3 is comparable to Figure 2.1. In Figure 2.3, relatively high productivity growth in the 1st sector results in decreasing effects of the 2nd commodity price. Its way of effectiveness is compounded with cyclical motions and trends.

Conversely in Figures 2.4–2.5, the 2nd sector's productivity grows rapidly. In these cases, basically the same results are obtained as in Figure 2.3.

Lastly Figure 2.6 shows extraordinarily high productivity growth in the 1st sector. In the case of $s_1 = -0.9$, we can get the results as reported in Figure 2.6, but for the range of values between $s_1 = -0.4$ and $s_1 = -0.8$, our calculations overflow.

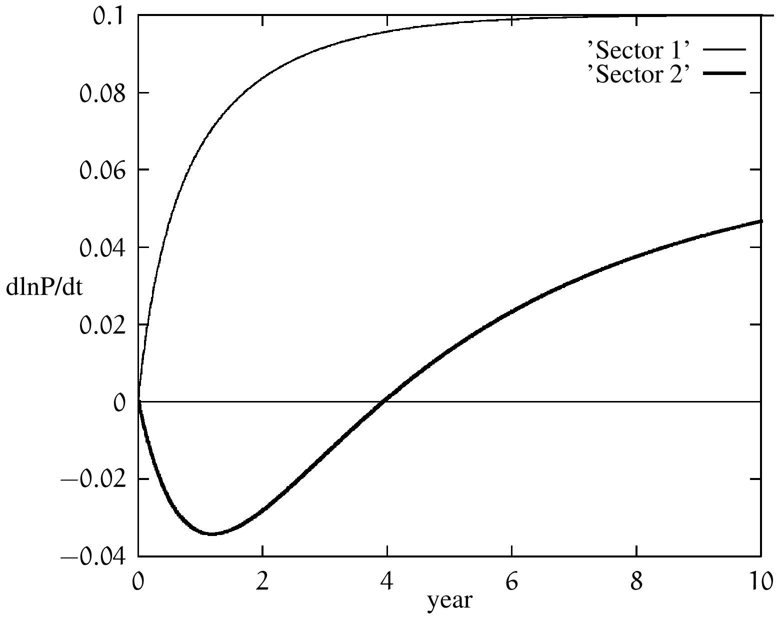
Detailed results about whether the calculations overflow or not are also given. Those diagrams show the likelihood of instability of the system with different parameters values for the cost shares. The two potential sources of overflow are as follows. One possible source is dynamic instability which depends on the characteristic roots of the determinant D. The other source is the numerical approximation method used which may cause overflow if the differential equation is a stiff equation in the sense that it has both large and small characteristic roots. The former case displays a dynamic property of productivity growth, and the latter case shows the difficulty when we treat continuous time variables as discrete time. In our numerical experiments, I first set dt as 1/100 year and then in the diverging cases I set dt as 1/200, but the diverging properties did not improve. Therefore, for reasonably small time units, our results show the instability of the effects of productivity growth for the whole economic system incorporating capital gains. But these characteristics are more fully investigated in the nect section. [2]In the divergent case, the sign of determinant D is negative and its absolute value become relatively large(but less than 1).



List of parameters in Figure 1

$a_{11} = 0.610$	$a_{12} = 0.190$	$a_{21} = 0.095$	$a_{22} = 0.128$
$b_{11} = 0.095$	$b_{12} = 0.172$	$b_{21} = 0.050$	$b_{22} = 0.240$
$r_{11} = 0.100$	$r_{12} = 0.140$	$r_{21} = 0.080$	$r_{22} = 0.100$
$w_{11} = 0.705$	$w_{12} = 0.362$	$w_{21} = 0.145$	$w_{22} = 0.368$
$s_1 = -0.010$	$s_2 = 0.010$		

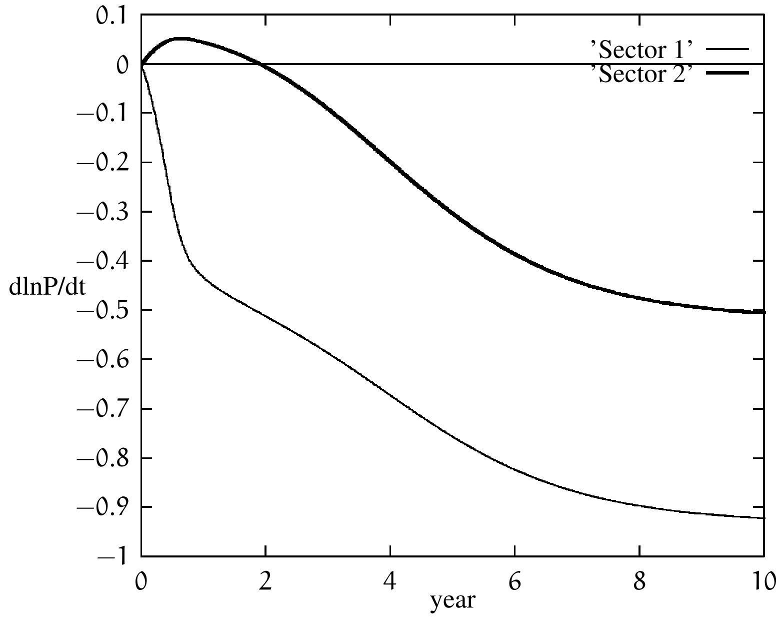
Figure 2.1: The change rate of the prices



List of parameters in Figure 2

$a_{11} = 0.610$	$a_{12} = 0.190$	$a_{21} = 0.095$	$a_{22} = 0.128$
$b_{11} = 0.095$	$b_{12} = 0.172$	$b_{21} = 0.050$	$b_{22} = 0.240$
$r_{11} = 0.100$	$r_{12} = 0.140$	$r_{21} = 0.080$	$r_{22} = 0.100$
$w_{11} = 0.705$	$w_{12} = 0.362$	$w_{21} = 0.145$	$w_{22} = 0.368$
$s_1 = 0.100$	$s_2 = 0.010$		

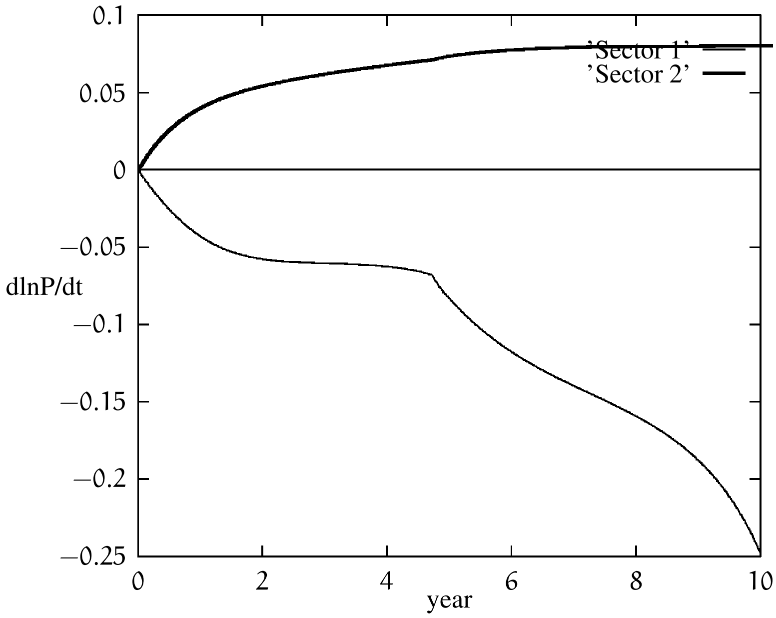
Figure 2.2: The change rate of the prices



List of parameters in Figure 3

$a_{11} = 0.610$	$a_{12} = 0.190$	$a_{21} = 0.095$	$a_{22} = 0.128$
$b_{11} = 0.095$	$b_{12} = 0.172$	$b_{21} = 0.050$	$b_{22} = 0.240$
$r_{11} = 0.100$	$r_{12} = 0.140$	$r_{21} = 0.080$	$r_{22} = 0.100$
$w_{11} = 0.705$	$w_{12} = 0.362$	$w_{21} = 0.145$	$w_{22} = 0.368$
$s_1 = -0.200$	$s_2 = 0.010$		

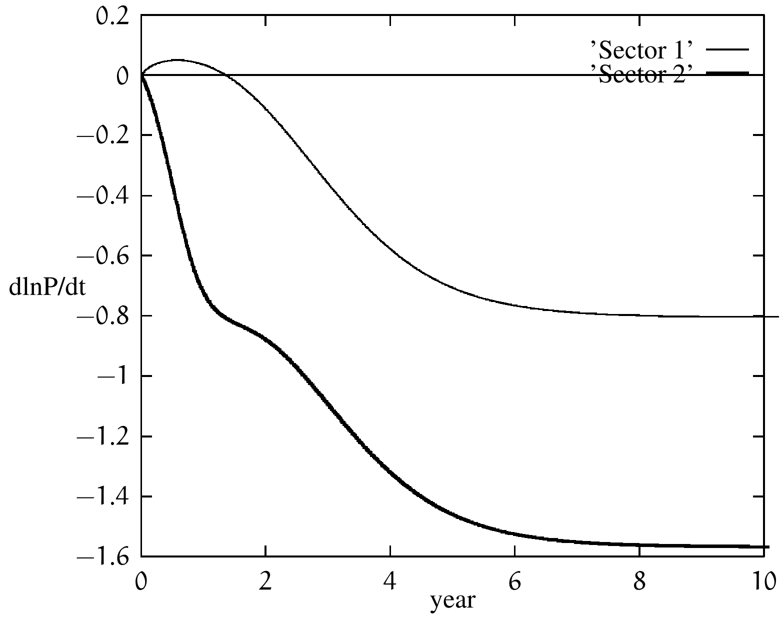
Figure 2.3: The change rate of the prices



List of parameters in Figure 4

$a_{11} = 0.610$	$a_{12} = 0.190$	$a_{21} = 0.095$	$a_{22} = 0.128$
$b_{11} = 0.095$	$b_{12} = 0.172$	$b_{21} = 0.050$	$b_{22} = 0.240$
$r_{11} = 0.100$	$r_{12} = 0.140$	$r_{21} = 0.080$	$r_{22} = 0.100$
$w_{11} = 0.705$	$w_{12} = 0.362$	$w_{21} = 0.145$	$w_{22} = 0.368$
$s_1 = -0.010$	$s_2 = 0.100$		

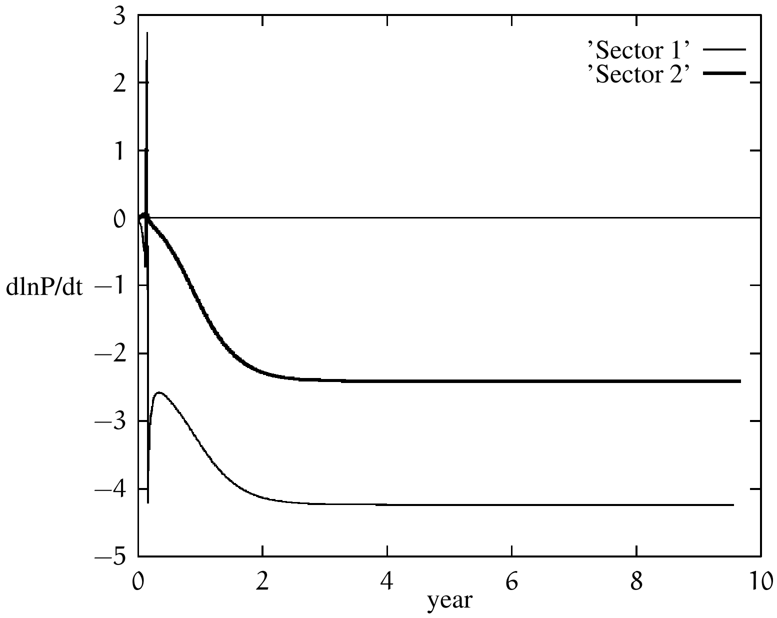
Figure 2.4: The change rate of the prices



List of parameters in Figure 5

$a_{11} = 0.610$	$a_{12} = 0.190$	$a_{21} = 0.095$	$a_{22} = 0.128$
$b_{11} = 0.095$	$b_{12} = 0.172$	$b_{21} = 0.050$	$b_{22} = 0.240$
$r_{11} = 0.100$	$r_{12} = 0.140$	$r_{21} = 0.080$	$r_{22} = 0.100$
$w_{11} = 0.705$	$w_{12} = 0.362$	$w_{21} = 0.145$	$w_{22} = 0.368$
$s_1 = -0.010$	$s_2 = -0.700$		

Figure 2.5: The change rate of the prices



List of parameters in Figure 6

$a_{11} = 0.610$	$a_{12} = 0.190$	$a_{21} = 0.095$	$a_{22} = 0.128$
$b_{11} = 0.095$	$b_{12} = 0.172$	$b_{21} = 0.050$	$b_{22} = 0.240$
$r_{11} = 0.100$	$r_{12} = 0.140$	$r_{21} = 0.080$	$r_{22} = 0.100$
$w_{11} = 0.705$	$w_{12} = 0.362$	$w_{21} = 0.145$	$w_{22} = 0.368$
$s_1 = -0.900$	$s_2 = 0.010$		

Figure 2.6: The change rate of the prices

2.4 An autonomous system of the price propagation

In this section, an autonomous system of the price equation (2.17) is investigated. For simplicity, assume that all the cost shares are constant through this section.

Assumption: Constant cost shares The cost shares $\mathbf{S_X}$, $\mathbf{S_K}$, $\mathbf{S_L}$ are constant over time.

$$\mathbf{Sr}\frac{d^2 \ln \mathbf{p}}{dt^2} + (\mathbf{I} - \mathbf{S_X} - \mathbf{S_K})\frac{d \ln \mathbf{p}}{dt} = 0. \tag{2.17}$$

This is, in fact, a first order differential equation system for the rate of price changes. The system is non-linear, since $\mathbf{Sr}$ depends on $d \ln \mathbf{p}/dt$. Let $\mathbf{z}$ denote $d \ln \mathbf{p}/dt$ for simplicity, and the system is expressed as follows:

$$\begin{aligned} \mathbf{Sr}(\mathbf{z})\frac{d\mathbf{z}}{dt} &= -(\mathbf{I} - \mathbf{S_X} - \mathbf{S_K})\,\mathbf{z}, \\ \frac{d \ln \mathbf{p}}{dt} &= \mathbf{z} \end{aligned} \tag{2.18}$$

There are two special cases to be highlighted. One of them is the singular matrix $\mathbf{Sr}(\mathbf{z})$. If the matrix $\mathbf{Sr}(\mathbf{z})$ is singular, the system cannot describe price changes. The other case is that the price change of a sector z_i is equal to $r_j + \delta_{ij}$. In this latter case, one of the elements of the matrix $\mathbf{Sr}(\mathbf{z})$ becomes infinitely positive or infinitely negative. The behaviour of the price changes around these points may change drastically.

Except for these two cases, the system has a fixed point $\mathbf{z} = \mathbf{0}$. And the behaviour of the price changes around the fixed point can be described using the eigenvalues of the following Jacobian matrix of first partial derivatives at $\mathbf{z} = \mathbf{0}$.

$$\mathbf{J}_\alpha(\mathbf{z}) = -D_z\left(\mathbf{Sr}(\mathbf{z})^{-1}(\mathbf{I} - \mathbf{S_X} - \mathbf{S_K})\,\mathbf{z}\right), \tag{2.19}$$

where D_z denotes a vector partial derivative operator $[\partial/\partial z_j]$ that operates a vector valued function.

$\mathbf{J}_\alpha$ with parameters α_0 may have a zero eigenvalue at the fixed point $\mathbf{z} = \mathbf{0}$, the point $(\mathbf{0}, \alpha_0)$ should be a bifurcation point. But the following discussion shows that this system does not have a bifurcation phenomenon. The matrix $\mathbf{Sr}(\mathbf{z})^{-1}(\mathbf{I}-\mathbf{S_X}-\mathbf{S_K})$ can be expressed by $\mathbf{W}(\mathbf{z})$ in general.

$$\boldsymbol{W}(z)z = \begin{pmatrix} \sum_{j=1}^{n} W_{1j}(z)z_j \\ \sum_{j=1}^{n} W_{2j}(z)z_j \\ \vdots \\ \sum_{j=1}^{n} W_{nj}(z)z_j \end{pmatrix}$$

$$D_z(\boldsymbol{W}(z)z) = D_z \begin{pmatrix} \sum_{j=1}^{n} W_{1j}(z)z_j \\ \sum_{j=1}^{n} W_{2j}(z)z_j \\ \vdots \\ \sum_{j=1}^{n} W_{nj}(z)z_j \end{pmatrix}$$

$$D_z(\boldsymbol{W}(z)z) = \begin{pmatrix} \sum_{j=1}^{n} \frac{\partial W_{1j}}{\partial z_1} z_j + W_{11} & \sum_{j=1}^{n} \frac{\partial W_{1j}}{\partial z_2} z_j + W_{12} & \dots & \sum_{j=1}^{n} \frac{\partial W_{2j}}{\partial z_n} z_j + W_{1n} \\ \sum_{j=1}^{n} \frac{\partial W_{2j}}{\partial z_1} z_j + W_{21} & \sum_{j=1}^{n} \frac{\partial W_{2j}}{\partial z_2} z_j + W_{22} & \dots & \sum_{j=1}^{n} \frac{\partial W_{2j}}{\partial z_n} z_j + W_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \sum_{j=1}^{n} \frac{\partial W_{nj}}{\partial z_1} z_j + W_{n1} & \sum_{j=1}^{n} \frac{\partial W_{nj}}{\partial z_2} z_j + W_{n2} & \dots & \sum_{j=1}^{n} \frac{\partial W_{nj}}{\partial z_n} z_j + W_{nn} \end{pmatrix}$$

Thus the Jacobian matrix at $z=0$ is

$$\begin{aligned} D_z(\boldsymbol{W}(z)z)|_{z=0} &= \begin{pmatrix} W_{11}(0) & W_{12}(0) & \dots & W_{1n}(0) \\ W_{21}(0) & W_{22}(0) & \dots & W_{2n}(0) \\ \vdots & \vdots & \dots & \vdots \\ W_{n1}(0) & W_{n2}(0) & \dots & W_{nn}(0) \end{pmatrix} \\ &= \mathbf{Sr}(0)^{-1}(\mathbf{I} - \mathbf{S_X} - \mathbf{S_K}). \end{aligned} \tag{2.20}$$

The determinant of a product of two $n \times n$ matrices is a product of two determinants of $n \times n$ matrices suggests that

$$|D_z(\boldsymbol{W}(z)z)|_{z=0}| = |\mathbf{Sr}(0)^{-1}||\mathbf{I} - \mathbf{S_X} - \mathbf{S_K}|. \tag{2.21}$$

The determinants $|\mathbf{I} - \mathbf{S_X} - \mathbf{S_K}|$ does not become zero because of the Hawkins-Simon's condition, and the determinants $|\mathbf{Sr}(0)^{-1}|$ is a reciprocal of $|\mathbf{Sr}(0)|$. $|\mathbf{Sr}(0)|$ is possible to be zero, but not infinitely large, unless both the interest rate and the depreciation rate are equal to zero.

2.4.1 Two-sector autonomous model

To illustrate the eigenvalues of the Jacobian matrix $\mathbf{J}_\alpha(z)$, I derived the two sector system explicitly. The equation system can be described as follows:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = -\begin{pmatrix} \frac{S_{K11}}{r_1+\delta_{11}-z_1} & \frac{S_{K21}}{r_1+\delta_{21}-z_2} \\ \frac{S_{K12}}{r_2+\delta_{12}-z_1} & \frac{S_{K22}}{r_2+\delta_{22}-z_2} \end{pmatrix}^{-1} \times \begin{pmatrix} 1-S_{X11}-S_{K11} & -S_{X21}-S_{K21} \\ -S_{X12}-S_{K12} & 1-S_{X22}-S_{K22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{2.22}$$

or

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = -\frac{1}{D(z)} \begin{pmatrix} \left(\frac{(1-S_{X11}-S_{K11})S_{K22}}{r_2+\delta_{22}-z_2} + \frac{(S_{X12}+S_{K12})S_{K21}}{r_1+\delta_{21}-z_2}\right) z_1 - \left(\frac{(1-S_{X22}-S_{K22})S_{K21}}{r_1+\delta_{21}-z_2} + \frac{(S_{X21}+S_{K21})S_{K22}}{r_2+\delta_{22}-z_2}\right) z_2 \\ -\left(\frac{(1-S_{X11}-S_{K11})S_{K12}}{r_2+\delta_{12}-z_1} + \frac{(S_{X12}+S_{K12})S_{K11}}{r_1+\delta_{11}-z_1}\right) z_1 + \left(\frac{(1-S_{X22}-S_{K22})S_{K21}}{r_1+\delta_{11}-z_1} + \frac{(S_{X21}+S_{K21})S_{K12}}{r_2+\delta_{12}-z_1}\right) z_2 \end{pmatrix}, \tag{2.23}$$

where D is the determinant of $\mathbf{Sr}$,

$$D(z) = \frac{S_{K11}S_{K22}}{(r_1+\delta_{11}-z_1)(r_2+\delta_{22}-z_2)} - \frac{S_{K12}S_{K21}}{(r_1+\delta_{21}-z_2)(r_2+\delta_{12}-z_1)}.$$

2.4.2 Behaviour around the fixed point z=0

Equation (2.23) is still complicated to calculate the Jacobian matrix, but substituting z=0 into the Jacobian matrix makes all the derivatives related with fractions disappear.

$$\mathbf{J}_\alpha(0) = \frac{1}{D(0)} \begin{pmatrix} -\left(\frac{(1-S_{X11}-S_{K11})S_{K22}}{r_2+\delta_{22}} + \frac{(S_{X12}+S_{K12})S_{K21}}{r_1+\delta_{21}}\right) & \left(\frac{(1-S_{X22}-S_{K22})S_{K21}}{r_1+\delta_{21}} + \frac{(S_{X21}+S_{K21})S_{K22}}{r_2+\delta_{22}}\right) \\ \left(\frac{(1-S_{X11}-S_{K11})S_{K12}}{r_2+\delta_{12}} + \frac{(S_{X12}+S_{K12})S_{K11}}{r_1+\delta_{11}}\right) & -\left(\frac{(1-S_{X22}-S_{K22})S_{K11}}{r_1+\delta_{11}} + \frac{(S_{X21}+S_{K21})S_{K12}}{r_2+\delta_{12}}\right) \end{pmatrix}. \tag{2.24}$$

$D(0)$ is defined as

$$D(0) = \frac{S_{K11}S_{K22}}{(r_1+\delta_{11})(r_2+\delta_{22})} - \frac{S_{K12}S_{K21}}{(r_1+\delta_{21})(r_2+\delta_{12})}. \tag{2.25}$$

The system has two real eigenvalues due to the fact that product of the off diagonal elements of the Jacobian matrix $\mathbf{J}_\alpha(0)$ is positive.[3]

It can be shown that the determinant of $\mathbf{J}_\alpha(0)$ is

$$\begin{aligned}
|\mathbf{J}_\alpha(0)| &= \frac{1}{D(0)^2}\Bigg\{\left(\frac{(1-S_{X11}-S_{K11})S_{K22}}{r_2+\delta_{22}}+\frac{(S_{X12}+S_{K12})S_{K21}}{r_1+\delta_{21}}\right)\\
&\quad\times\left(\frac{(1-S_{X22}-S_{K22})S_{K11}}{r_1+\delta_{11}}+\frac{(S_{X21}+S_{K21})S_{K12}}{r_2+\delta_{12}}\right)\\
&\quad-\left(\frac{(1-S_{X22}-S_{K22})S_{K21}}{r_1+\delta_{21}}+\frac{(S_{X21}+S_{K21})S_{K22}}{r_2+\delta_{22}}\right)\\
&\quad\times\left(\frac{(1-S_{X11}-S_{K11})S_{K12}}{r_2+\delta_{12}}+\frac{(S_{X12}+S_{K12})S_{K11}}{r_1+\delta_{11}}\right)\Bigg\}\\
&= \frac{1}{D(0)}|\mathbf{A}|
\end{aligned}$$

where

$$\begin{aligned}
|\mathbf{A}| &= (1-S_{X11}-S_{K11})(1-S_{X22}-S_{K22})\\
&\quad -(S_{X12}+S_{K12})(S_{X21}+S_{K21}).
\end{aligned} \tag{2.26}$$

Thus, the sign of the determinant $|\mathbf{J}_\alpha(0)|$ depends on the sign of the determinant $D(0)$, while $|\mathbf{A}|$ is positive because of the Hawkins-Simon's condition[4].

The following classification of the system can be obtained.

1. If $|\mathbf{J}_\alpha(0)| > 0$, *i.e.* $D(0) > 0$ the system has a fixed point of stable.
2. If $|\mathbf{J}_\alpha(0)| < 0$, *i.e.* $D(0) < 0$ the system has a fixed point of saddle.
3. The eigenvalues are diverge, when determinant $D(0)$ is zero.

---

[3]The eigenvalues of a 2×2 matrix $\mathbf{A}$ can be derived as follows: λ denotes an eigenvalue of $\mathbf{A}$.

$$\begin{aligned}
|\lambda\mathbf{I}-\mathbf{A}| &= \begin{vmatrix} \lambda-a_{11} & -a_{12} \\ -a_{21} & \lambda-a_{22} \end{vmatrix}\\
&= \lambda^2-(a_{11}+a_{22})\lambda+a_{11}a_{22}-a_{12}a_{21}=0.
\end{aligned}$$

The determinant D of the second order equation for λ is

$$\begin{aligned}
D &= (a_{11}+a_{22})^2-4(a_{11}a_{22}-a_{12}a_{21})\\
&= (a_{11}-a_{22})^2+4a_{12}a_{21}.
\end{aligned}$$

Thus $a_{12}a_{21} < 0$ is necessary for eigenvalues λ to have imaginary parts.

[4]The Hawkins-Simon's condition for a two sector input-output model is as follows, where $\mathbf{A}$ is a input-coefficient matrix:

$$\begin{vmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{vmatrix} > 0.$$

The condition implies the system operates in positive production for all the sectors. In this case, the inputs are not only material inputs S_{Xij} but also includes capital goods S_{Kij}. The sum of both inputs needs to be in production possibility.

$D(0) > 0$ can be rewrite from (2.25) as:

$$\frac{S_{K11}S_{K22}}{S_{K12}S_{K21}} > \frac{(r_1+\delta_{11})(r_2+\delta_{22})}{(r_1+\delta_{21})(r_2+\delta_{12})}.$$

The left hand side of the inequality means that relatively larger input coefficients of their own capital goods than those of the other sector's capital goods, and the right hand side of the inequality shows that the price of their own secotr's capital goods is relatively smaller than the price of the other sector's capital goods. That is, given the relative cost, if the economic system starts to rely more heavily on outsourced capital goods than before, the system may hit a bifurcation point that shows saddle point instability. Substitute the definition of S_{Kij} into the condition, and it yields the condition in terms of capital goods quantity:

$$\frac{k_{11}}{k_{21}} > \frac{k_{12}}{k_{22}}.$$

This again shows that the relatively large own capital input implies saddle point instability of the system.[5]

The previous classification can be interpreted as follows.

1. If $\frac{S_{K11}S_{K22}}{S_{K12}S_{K21}} > \frac{(r_1+\delta_{11})(r_2+\delta_{22})}{(r_1+\delta_{21})(r_2+\delta_{12})}$ or $\frac{k_{11}}{k_{21}} > \frac{k_{12}}{k_{22}}$, the system has a stable fixed point.

2. If $\frac{S_{K11}S_{K22}}{S_{K12}S_{K21}} < \frac{(r_1+\delta_{11})(r_2+\delta_{22})}{(r_1+\delta_{21})(r_2+\delta_{12})}$ or $\frac{k_{11}}{k_{21}} < \frac{k_{12}}{k_{22}}$, the system has a fixed point of saddle.

3. If $\frac{S_{K11}S_{K22}}{S_{K12}S_{K21}} = \frac{(r_1+\delta_{11})(r_2+\delta_{22})}{(r_1+\delta_{21})(r_2+\delta_{12})}$ or $\frac{k_{11}}{k_{21}} = \frac{k_{12}}{k_{22}}$, the eigenvalue of the system is diverged.

2.4.3 Checking another fixed points $\dot{z} = 0$

Set (2.23) equal to 0, there may be another fixed point in the system. Similar discussions to the Jacobian matrix at $z = 0$, that the matrix $\mathbf{Sr}(z)^{-1}(\mathbf{I} - \mathbf{S_X} - \mathbf{S_K})$ must be singular to hold the equilibrium with $z \neq 0$. Because of $|\mathbf{I} - \mathbf{S_X} - \mathbf{S_K}| > 0$, it should be $\mathbf{Sr}(z)^{-1} = 0$. This is impossible unless $\mathbf{Sr}(z)$ is singular. I shall show next that there are many singular points.

[5] Benhabib and Nishimura [1998] assume single interest rate and single depreciation rate, but introduce externality. They obtain indeterminacy, i.e. multiple equilibria in the two sector model.

Nonetheless it is necessary to solve the equilibrium equations to show the phase portraits of the systems. Solve the next equations for z_1 and z_2:

$$\begin{aligned} \dot{z}_1 &= 0 \\ \dot{z}_2 &= 0 \end{aligned} \tag{2.27}$$

That is

$$\begin{aligned} &\left(\frac{(1-S_{11}-S_{K11})S_{K22}}{r_2+\delta_{22}-z_2} + \frac{(S_{12}+S_{K12})S_{K21}}{r_1+\delta_{21}-z_2}\right) z_1 \\ &-\left(\frac{(1-S_{22}-S_{K22})S_{K21}}{r_1+\delta_{21}-z_2} + \frac{(S_{21}+S_{K21})S_{K22}}{r_2+\delta_{22}-z_2}\right) z_2 = 0 \\ &-\left(\frac{(1-S_{11}-S_{K11})S_{K12}}{r_2+\delta_{12}-z_1} + \frac{(S_{12}+S_{K12})S_{K11}}{r_1+\delta_{11}-z_1}\right) z_1 \\ &+\left(\frac{(1-S_{22}-S_{K22})S_{K11}}{r_1+\delta_{11}-z_1} + \frac{(S_{21}+S_{K21})S_{K12}}{r_2+\delta_{12}-z_1}\right) z_2 = 0 \end{aligned} \tag{2.28}$$

There are two asymptotes in each equation. One of these is a vertical or horizontal line.

$$\begin{aligned} z_2 &= \frac{(1-S_{11}-S_{K11})S_{K22}(r_1+\delta_{21})+(S_{12}+S_{K12})S_{K21}(r_2+\delta_{22})}{(1-S_{11}-S_{K11})S_{K22}+(S_{12}+S_{K12})S_{K21}}, \\ &\qquad \text{for } \frac{dz_1}{dt} = 0\ . \\ z_1 &= \frac{(1-S_{22}-S_{K22})S_{K11}(r_2+\delta_{12})+(S_{21}+S_{K21})S_{K12}(r_1+\delta_{11})}{(1-S_{22}-S_{K22})S_{K11}+(S_{21}+S_{K21})S_{K12}}, \\ &\text{for } \frac{dz_2}{dt} = 0 \end{aligned}$$

The other is a straight line of which the gradient does not depend on the interest rate or the depreciation rate. However the line is too complicated to describe in terms of the original parameters in the system S_{ij} or $S_K ij$. It can be shown that equation (2.28) can be arranged into the following form:

$$\begin{aligned} z_1 &= \frac{b_1}{d_1}z_2 - \frac{a_1d_1-b_1c_1}{d_1{}^2} + \frac{(a_1d_1-b_1c_1)c_1}{d_1{}^2(c_1-d_1z_2)} : \quad \text{for } \frac{dz_1}{dt} = 0 \\ z_2 &= \frac{b_2}{d_2}z_2 - \frac{a_2d_2-b_2c_2}{d_2{}^2} + \frac{(a_2d_2-b_2c_2)c_2}{d_2{}^2(c_2-d_2z_2)} : \quad \text{for } \frac{dz_2}{dt} = 0, \end{aligned}$$

where we temporally introduce the parameters a_i, b_i, c_i, d_i $(i = 1, 2)$.

$$\begin{aligned} a_1 &= (S_{21} + S_{K21})S_{K22}(r_1 + \delta_{21}) \\ &\quad +(1 - S_{22} - S_{K22})S_{K21}(r_2 + \delta_{22}) \\ b_1 &= (S_{21} + S_{K21})S_{K22} + (1 - S_{22} - S_{K22})S_{K21} \\ c_1 &= (S_{12} + S_{K12})S_{K21}(r_2 + \delta_{22}) \\ &\quad +(1 - S_{11} - S_{K11})S_{K22}(r_1 + \delta_{21}) \\ d_1 &= (S_{12} + S_{K12})S_{K21} + (1 - S_{11} - S_{K11})S_{K22} \\ a_2 &= (S_{12} + S_{K12})S_{K11}(r_2 + \delta_{12}) \\ &\quad +(1 - S_{11} - S_{K11})S_{K12}(r_1 + \delta_{11}) \\ b_2 &= (S_{12} + S_{K12})S_{K11} + (1 - S_{11} - S_{K11})S_{K12} \\ c_2 &= (S_{21} + S_{K21})S_{K12}(r_1 + \delta_{11}) \\ &\quad +(1 - S_{22} - S_{K22})S_{K11}(r_2 + \delta_{12}) \\ d_2 &= (S_{21} + S_{K21})S_{K12} + (1 - S_{22} - S_{K22})S_{K11}. \end{aligned}$$

If $a_i d_i - b_i c_i = 0$, the second type of asymptotes disappears. Equation $dz_i/dt = 0$ becomes a straight line through the origin, and its gradient does not depend on the interest rate or the depreciation rate. The gradient is determined by the constants (here I assume that they are technological factors) S_{ij}, and S_{Kij}. Thus, the determinants $a_i d_i - b_i c_i = 0$ are other important factors in the system, and it can be shown as:

$$\begin{aligned} a_1 d_1 - b_1 c_1 &= (r_2 - r_1 + \delta_{22} - \delta_{21}) S_{K21} S_{K22} |\mathbf{A}| \\ a_2 d_2 - b_2 c_2 &= (r_1 - r_2 + \delta_{11} - \delta_{12}) S_{K11} S_{K12} |\mathbf{A}|, \end{aligned}$$

where

$$\begin{aligned} |\mathbf{A}| &= (1 - S_{11} - S_{K11})(1 - S_{22} - S_{K22}) \\ &\quad -(S_{12} + S_{K12})(S_{21} + S_{K21}). \end{aligned}$$

The sign of $|\mathbf{A}|$ is again positive because of the Hawkins-Simon's condition in terms of the cost share, and each S_{Kij} is positive. The sign of $a_i d_i - b_i c_i = 0$ is determined by the magnitude of the interest rate and the depreciation rate.

2.4.4 Classification of the autonomous system

Summary of the above discussions provides the general classification of the two sector system. First, the system is stable at the fixed point, or saddle at the fixed point. Second, the sign of intercept of the asymptote for each equation $\dot{z}_1 = 0$ and $\dot{z}_2 = 0$, there are four cases. The other factor that is not considered here is the magnitude of gradient of the asymptote.

There are at most four set of solutions for z_1 and z_2 including $\mathbf{0}$.The other solutions are from the 3rd order polynomial equation, and the determinant vanishes at the points. This situation can be occurred at the points, where three curves intersect in Figures 2.4.4–2.4.4.

Case I The system is stable at the fixed point $\mathbf{z} = \mathbf{0}$. The two asymptotes for $\dot{z}_1 = 0$ and $\dot{z}_2 = 0$ both have positive intercepts to the other axis: $a_1 d_1 - b_1 c_1 > 0$, $a_2 d_2 - b_2 c_2 > 0$. An economic meaning of these conditions is that the nominal cost (interest rate and depreciation) of capital goods is higher in the own sector's investment that the other sector's.

Case II The system is stable at the fixed point $\mathbf{z} = \mathbf{0}$. The two asymptotes for $\dot{z}_1 = 0$ and $\dot{z}_2 = 0$ both have negative intercepts to the other axis: $a_1 d_1 - b_1 c_1 < 0$, $a_2 d_2 - b_2 c_2 < 0$. The nominal cost (interest rate and depreciation) of capital goods is lower in the own sector's investment that the other sector's.

Case III The system is stable at the fixed point $z = 0$. The asymptote for $\dot{z}_1 = 0$ has positive intercepts to z_2 axis: $a_1 d_1 - b_1 c_1 > 0$. The asymptote for $\dot{z}_2 = 0$ has negative intercepts to z_1 axis: $a_2 d_2 - b_2 c_2 < 0$. The nominal cost (interest rate and depreciation) of capital goods is higher in the own sector's investment that the other sector's for the commodity 2. The nominal cost (interest rate and depreciation) of capital goods is lower in the own sector's investment that the other sector's for the commodity 1.

Case IV The system is stable at the fixed point $z = 0$. The asymptote for $\dot{z}_1 = 0$ has negative intercepts to z_2 axis: $a_1 d_1 - b_1 c_1 < 0$. The asymptote for $\dot{z}_2 = 0$ has positive intercepts to z_1 axis: $a_2 d_2 - b_2 c_2 > 0$. The nominal cost (interest rate and depreciation) of capital goods is lower in the own sector's investment that the other sector's for the commodity 2. The nominal cost (interest rate and depreciation) of capital goods is higher in the own sector's investment that the other sector's for the commodity 1.

The same classification can be applicable to the system of saddle point.

Case V The system has a saddle point at the fixed point $z = 0$. The two asymptotes for $\dot{z}_1 = 0$ and $\dot{z}_2 = 0$ both have positive intercepts to the other axis: $a_1 d_1 - b_1 c_1 > 0$, $a_2 d_2 - b_2 c_2 > 0$. An economic meaning of these conditions is as follows that the nominal cost (interest rate and depreciation) of capital goods is higher in the own sector's investment that the other sector's.

Case VI The system has a saddle point at the fixed point $z = 0$. The two asymptotes for $\dot{z}_1 = 0$ and $\dot{z}_2 = 0$ both have negative intercepts to the other axis: $a_1 d_1 - b_1 c_1 < 0$, $a_2 d_2 - b_2 c_2 < 0$. The nominal cost (interest rate and depreciation) of capital goods is lower in the own sector's investment that the other sector's.

Case VII The system has a saddle point at the fixed point $z = 0$. The asymptote for $\dot{z}_1 = 0$ has positive intercepts to z_2 axis: $a_1 d_1 - b_1 c_1 > 0$. The asymptote for $\dot{z}_2 = 0$ has negative intercepts to z_1 axis: $a_2 d_2 - b_2 c_2 < 0$. The nominal cost (interest rate and depreciation) of capital goods is higher in the own sector's investment that the other sector's for the commodity 2. The nominal cost (interest rate and depreciation) of capital goods is lower in the own sector's investment that the other sector's for the commodity 1.

Case VIII The system has a saddle point at the fixed point $z = 0$. The asymptote for $\dot{z}_1 = 0$ has negative intercepts to z_2 axis: $a_1 d_1 - b_1 c_1 < 0$. The asymptote for $\dot{z}_2 = 0$ has positive intercepts to z_1 axis: $a_2 d_2 - b_2 c_2 > 0$. The nominal cost (interest rate and depreciation) of capital goods is lower in their own sector's investment that the other sector's for the commodity 2. The nominal cost (interest rate and depreciation) of capital goods is higher in their own sector's investment than the other sector's for the commodity 1.

Parameter sets for the numerical illustration can be shown in Tables 2.1–2.2.

Note of Table 2.1:
This table shows that the system has a stable fixed point at $z = 0$

The difference between cases I–IV comes from the interest rate and the depreciation rate. The technological parameters are common for all cases.

$a_1 d_1 - b_1 c_1 = (r_2 - r_1 + \delta_{22} - \delta_{21}) S_{K21} S_{K22} |\mathbf{A}|$

$a_2 d_2 - b_2 c_2 = (r_1 - r_2 + \delta_{11} - \delta_{12}) S_{K11} S_{K12} |\mathbf{A}|$

$a_1 d_1 - b_1 c_1$ implies the difference of nominal cost of capital goods 2 between the own sector 2 and the sector 1.

$a_2 d_2 - b_2 c_2$ implies the difference of nominal cost of capital goods 1 between the own sector 1 and the sector 2.

$|J_\alpha(0)|$ denotes the Jacobian at the fixed point $z = 0$: positive means stable around the fixed point.

'Eigenvalues' are of the solution λ for the equation, $|\lambda \mathbf{I} - J_\alpha(0)| = 0$.

'Eigenvectors' are the associate vectors z^* with each eigenvalue, $J_\alpha(0) z^* = \lambda z^*$.

Note of Table 2.2:
This table shows that the system has a saddle point at $z = 0$

The difference between cases I–IV and cases V–VIII comes from the technological parameters, especially from S_K.

Cases V–VIII shows larger capital goods input from the other sectors than own sectors.

Out sourcing of capital goods implies saddle point instability, as I explained in the text. The difference between cases I–IV comes from the interest rate and the depreciation rate. The technological parameters are common for all cases.

$a_1 d_1 - b_1 c_1 = (r_2 - r_1 + \delta_{22} - \delta_{21}) S_{K21} S_{K22} |\mathbf{A}|$

$a_2 d_2 - b_2 c_2 = (r_1 - r_2 + \delta_{11} - \delta_{12}) S_{K11} S_{K12} |\mathbf{A}|$

$a_1 d_1 - b_1 c_1$ implies the difference of nominal cost of capital goods 2 between the own sector 2 and the sector 1.

$a_2 d_2 - b_2 c_2$ implies the difference of nominal cost of capital goods 1 between the own sector 1 and the sector 2.

$|J_\alpha(0)|$ denotes the Jacobian at the fixed point $z = 0$: positive means stable around the fixed point.

'Eigenvalues' are of the solution λ for the equation, $|\lambda \mathbf{I} - J_\alpha(0)| = 0$.

'Eigenvectors' are the associate vectors z^* with each eigenvalue, $J_\alpha(0) z^* = \lambda z^*$.

Table 2.1: Parameter sets for the stable fixed point at $z = 0$

	Case I $a_1d_1 - b_1c_1 > 0$ $a_2d_2 - b_2c_2 > 0$	Case II $a_1d_1 - b_1c_1 < 0$ $a_2d_2 - b_2c_2 < 0$	Case III $a_1d_1 - b_1c_1 > 0$ $a_2d_2 - b_2c_2 < 0$	Case IV $a_1d_1 - b_1c_1 < 0$ $a_2d_2 - b_2c_2 > 0$
S_{X11}	0.04	0.04	0.04	0.04
S_{X12}	0.29	0.29	0.29	0.29
S_{X21}	0.19	0.19	0.19	0.19
S_{X22}	0.20	0.20	0.20	0.20
S_{K11}	0.42	0.42	0.42	0.42
S_{K12}	0.12	0.12	0.12	0.12
S_{K21}	0.15	0.15	0.15	0.15
S_{K22}	0.40	0.40	0.40	0.40
r_1	0.05	0.05	0.03	0.06
r_2	0.05	0.05	0.06	0.03
δ_{11}	0.25	0.15	0.15	0.15
δ_{12}	0.15	0.25	0.15	0.15
δ_{21}	0.15	0.25	0.25	0.25
δ_{22}	0.25	0.15	0.25	0.25
$\lvert J_\alpha(0)\rvert$	0.0540706	0.01915	0.0283218	0.0302222
Eigenvalue	-1.22028	-0.529053	-0.71621	-0.747259
Eigenvector at $z = 0$	$\begin{pmatrix} -0.731968 \\ 0.681339 \end{pmatrix}$	$\begin{pmatrix} -0.726971 \\ 0.686668 \end{pmatrix}$	$\begin{pmatrix} -0.538918 \\ 0.842358 \end{pmatrix}$	$\begin{pmatrix} -0.59224 \\ 0.805762 \end{pmatrix}$
Eigenvalue	-0.0443101	-0.0361967	-0.039544	-0.0404441
Eigenvector at $z = 0$	$\begin{pmatrix} -0.615461 \\ -0.788167 \end{pmatrix}$	$\begin{pmatrix} -0.610981 \\ -0.791646 \end{pmatrix}$	$\begin{pmatrix} 0.627869 \\ 0.778319 \end{pmatrix}$	$\begin{pmatrix} 0.61659 \\ 0.787285 \end{pmatrix}$

Table 2.2: Parameter sets for the saddle point at $z = 0$

	Case V	Case VI	Case VII	Case VIII
	$a_1d_1 - b_1c_1 > 0$	$a_1d_1 - b_1c_1 < 0$	$a_1d_1 - b_1c_1 > 0$	$a_1d_1 - b_1c_1 < 0$
	$a_2d_2 - b_2c_2 > 0$	$a_2d_2 - b_2c_2 < 0$	$a_2d_2 - b_2c_2 < 0$	$a_2d_2 - b_2c_2 > 0$
S_{X11}	0.04	0.04	0.04	0.04
S_{X12}	0.29	0.29	0.29	0.29
S_{X21}	0.19	0.19	0.19	0.19
S_{X22}	0.20	0.20	0.20	0.20
S_{K11}	0.14	0.14	0.14	0.14
S_{K12}	0.32	0.32	0.32	0.32
S_{K21}	0.15	0.15	0.15	0.15
S_{K22}	0.10	0.10	0.10	0.10
r_1	0.05	0.05	0.03	0.06
r_2	0.05	0.05	0.06	0.03
δ_{11}	0.25	0.15	0.15	0.15
δ_{12}	0.15	0.25	0.15	0.15
δ_{21}	0.15	0.25	0.25	0.25
δ_{22}	0.25	0.15	0.25	0.25
$\lvert J_\alpha(0)\rvert$	-0.351	-1.99964	-0.648356	-0.589276
Eigenvalue	1.73559	8.77869	3.13194	2.87207
Eigenvector at $z = 0$	$\begin{pmatrix} 0.508159 \\ -0.861263 \end{pmatrix}$	$\begin{pmatrix} 0.502609 \\ -0.864514 \end{pmatrix}$	$\begin{pmatrix} 0.380055 \\ -0.924964 \end{pmatrix}$	$\begin{pmatrix} 0.358354 \\ -0.933586 \end{pmatrix}$
Eigenvalue	-0.202237	-0.227783	-0.207014	-0.205174
Eigenvector at $z = 0$	$\begin{pmatrix} -0.560947 \\ -0.827852 \end{pmatrix}$	$\begin{pmatrix} -0.566327 \\ -0.824181 \end{pmatrix}$	$\begin{pmatrix} -0.564695 \\ -0.8253 \end{pmatrix}$	$\begin{pmatrix} -0.540816 \\ -0.841141 \end{pmatrix}$

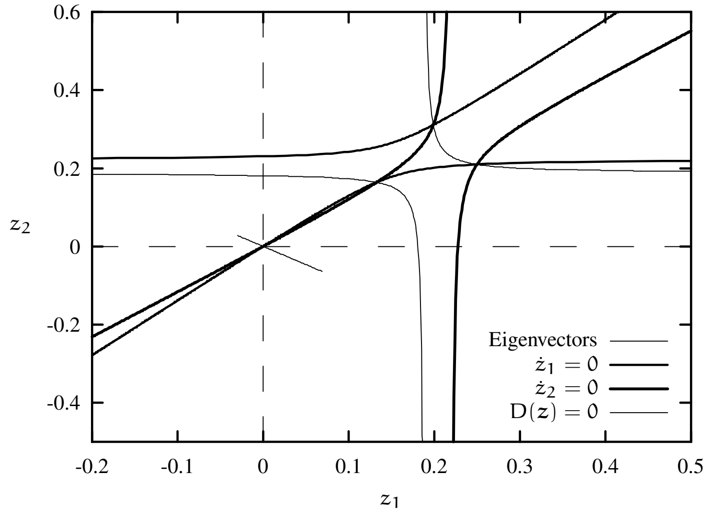


Figure 2.7: Case I: Stable fixed point at $z = 0$

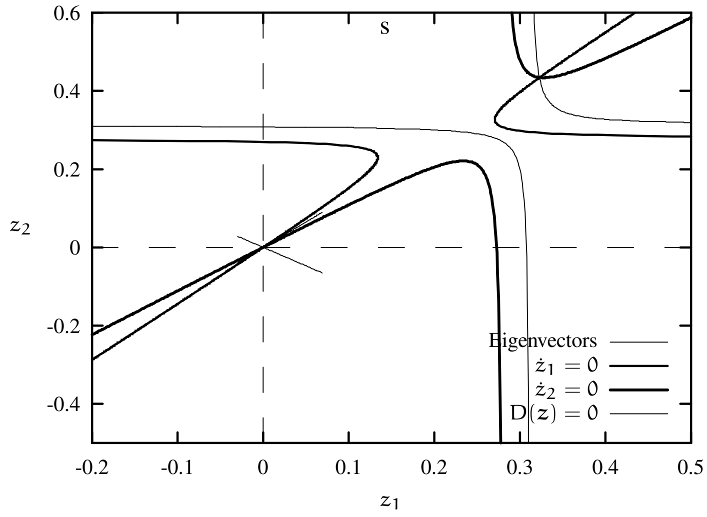


Figure 2.8: Case II: Stable fixed point at $z = 0$

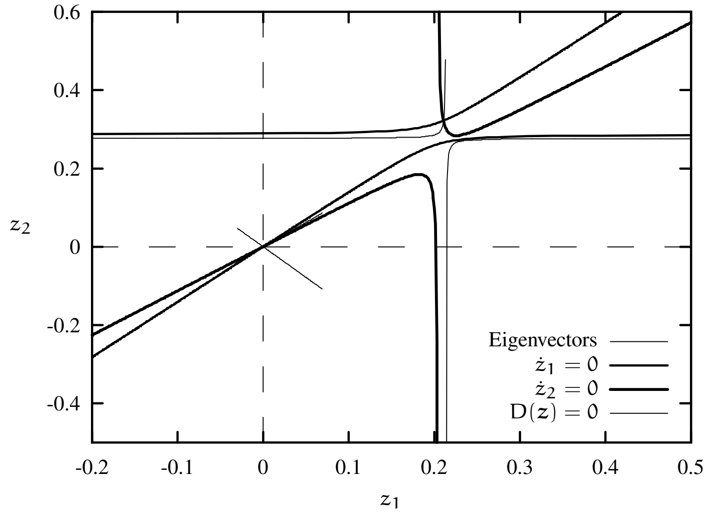


Figure 2.9: Case III: Stable fixed point at $z = 0$

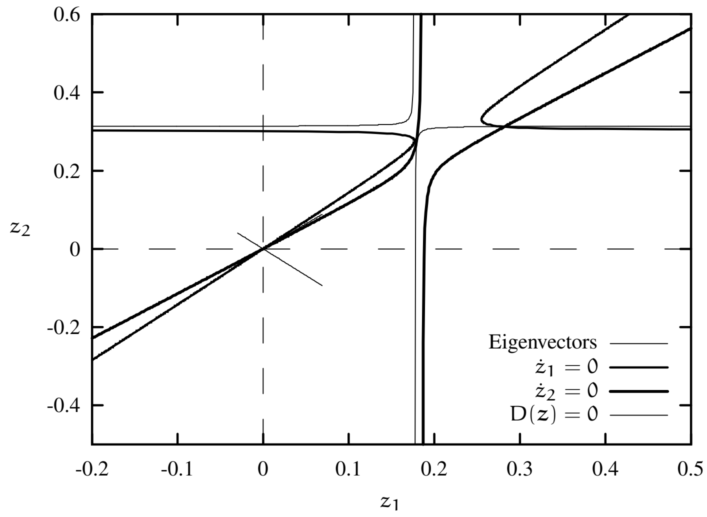


Figure 2.10: Case IV: Stable fixed point at $z = 0$

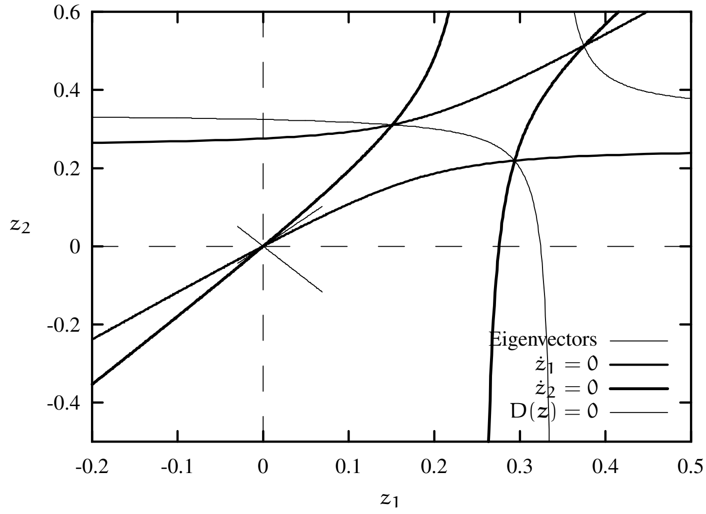


Figure 2.11: Case V: Saddle point at $z = 0$

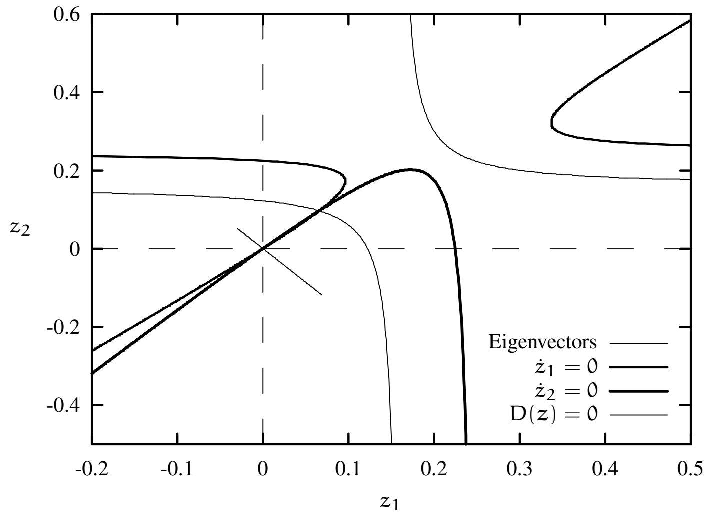


Figure 2.12: Case VI: Saddle point at $z = 0$

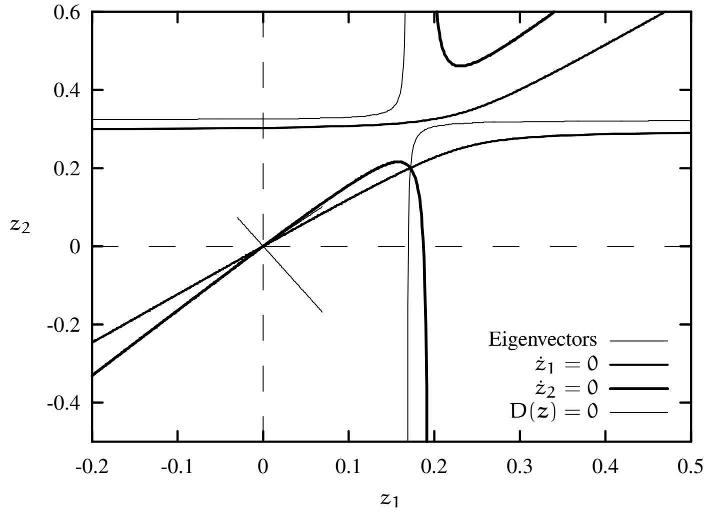


Figure 2.13: Case VII: Saddle point at $z = 0$

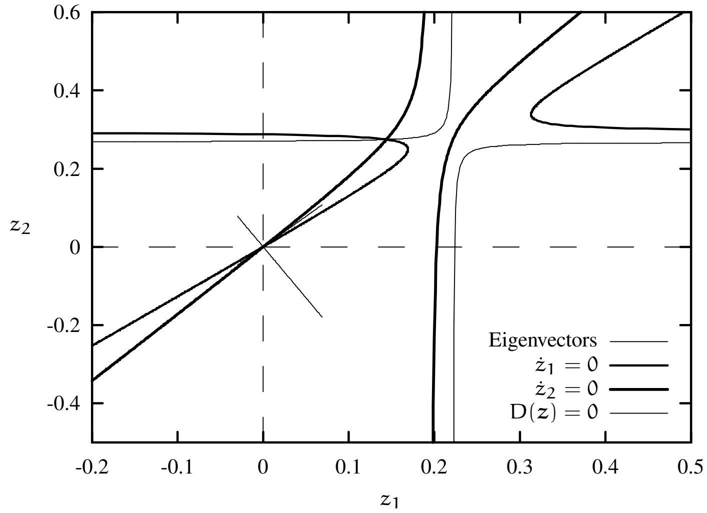


Figure 2.14: Case VIII: Saddle point at $z = 0$

Chapter 3

The stochastic system and its solutoin method

3.1 Random factor: an introdcution

This section surveys breafly how to introduce random factor into a system. I would like to present its conditions and to give a perspective for the generalisation. There are many introductory explanations by mathematitian, which reveal strict mathematical conditons. However, we cannot understand how economic variables and data phenomenologically relate with the conditions through the strict mathematical explanation.[1]. I am indebted to Reichl [1980] and van Kampen [1992] greatly, and fundamentally to the classical paper by Wang and Uhlenbeck [1945]. Here I would like to focus on fluctuations of the economic varialbes.

For example, I assume that the rate of change of the total factor productivity (TFP) obeys a stocastic process. The rate of change of TFP is a random process, which is accelerated or decelerated by number of sucesses and failures of technical progress. This is often assumed, but in fact it is only a working hypothesis. Nevertheless, considering that each improvement of technology is a very small step, but eventually these steps can be aggregated into an increase of TFP, hece we can apply the central limit theorem to the movement of TFP. That means that the distribution of the rate of change of TFP is assumed to be gaussian. In economics, mechanism of technical change is given in general, and normally treated as a black box.

[1]Among the explanations by mathematitioans, L. C. G. Rogers [1997] is an excelent overview of stochasitc processes in finance and easy to understand.

Economics does not analyse whether the current prevailing technology is superior or inferior to the past technologies, because it survives under the economic competition.

Such a formulation is quite common in biology. It is assumed that the gene frequency fluctuates under a stochastic process, when the biologist introduce a stochastic process into an evolution. A gene of some phenotype is assumed to be a mutually exclusive envent to a gene of the opposite phenotype. The model further assumes random mutation, and environmental facors (parameters) that affect one of the phenotypes profitable.[2] In this formulation, gene itself is not affected by natural selection, but the phenotype. Past environmental history determines the present phenotype, but we cannot know whether it is an evolution or not. Furthermore, even if there were the same environmental history as the past again, there is no reason that the same phenotype can survive. The model is a black box other than gene frequency. We cannot formulate multiple phenotypes as in the biological model, but we assumes that the level of TFP represents an adaptation to economic environment at each time and that it is enough to describe the survival of the fittest under the economic competition.

I would like to introduce such an economic stochastic process rather formally in this section. Let X is a stochastic variable, a stochastic process $Y_X(t) = f(X, t)$ at time t, and the probability with the value $Y_X(t) = y$ is described by the probability density function of X, $P_X(x)$, and Dirac's δ function as follows:

$$\begin{aligned} P_1(y,t) &= \int \delta(y - Y_x(t)) P_X(x) dx, \text{or} \\ &= <\delta(y - Y_x(t))>, \end{aligned} \tag{3.1}$$

where $Y_x(t) = f(x, t)$ is a sample function of the stochastic process. If $Y_X(t)$ has values y_1, y_2, ..., y_n at n points t_1, t_2, ..., t_n, then its joint density hierarchy function becomes as follows:

$$\begin{aligned} &P_n((y_1,t_1),(y_2,t_2),\ldots,(y_n,t_n)) \\ &\quad = \int \delta(y_1 - Y_x(t_1))\delta(y_2 - Y_x(t_2))\cdots\delta(y_n - Y_x(t_n))P_X(x)dx \end{aligned}$$

If a joint probability density hierarchy function P_n is satisfied with the following four conditions, and the mean exists, P_n can construct the stochastic process completely. This is the Kolmogorov existence theorem, and the proof is given by Billingsley [1995], for example.

1. $P_n \geq 0$

[2] See for example Kimura [1964].

2. P_n is the same value, even if (y_i, t_i) and (y_j, t_j) are exchanged.

3.

$$\int P_n((y_1,t_1),(y_2,t_2),\ldots,(y_n,t_n))dy_n = P_{n-1}((y_1,t_1),(y_2,t_2),\ldots,(y_{n-1},t_{n-1})) \quad (3.2)$$

4. $\int P_1(y,t)dy = 1$

In order to analyse concrete problems, we must introduce further assumptions, such as stationarity, Markov, or Gauss distribution. First I would like to show the limitation of stationarity. And second, I will introduce Markov and finally derive the Fokker-Planck equation.

3.1.1 Stationary process and its limitation

If a stochastic process is stationary, the probability density hierarchy function P_n remains unchanged as time passes to any τ.

$$\int P_n((y_1,t_1),(y_2,t_2),\ldots,(y_n,t_n)) = P_{n-1}((y_1,t_1+\tau),(y_2,t_2+\tau),\ldots,(y_n,t_n+\tau)) \quad (3.3)$$

In a stationary process, applying the Wiener-Khinchin theorem, and the power spectrum $I(\omega)$ can be obtained by Fourier transpose of its autocorrelation function. In a stationary process, autocorrelation function $\phi(t_1, t_2)$ is expressed by a fuction of time difference $\tau = t_2 - t_1$.

$$\begin{aligned} \phi(t_1,t_2) &= <(Y(t_1)-<Y(t_1)>)(Y(t_2)-<Y(t_2)>)> \\ &= <(Y(0)-<Y(0)>)(Y(\tau)-<Y(\tau)>)> \\ \phi(\tau) &= <(Y(0)-<Y(0)>)(Y(\tau)-<Y(\tau)>)> \quad (3.4) \end{aligned}$$

During $0 < t < T$ a sample function of a stochastic process is a normal function of t and it can be applicable to Fourier transformation. $\omega_n = \frac{2\pi n}{T}$ denotes the frequency. If Y_X is real number, the Fourier coefficient a_n has an restriction of $a_{-n} = a_n^*$.

$$Y_x(t) = \sum_{n=-\infty}^{\infty} a_n e^{i\omega_n t}$$

$$a_n = \frac{1}{T}\int_0^T Y_x(t)e^{-i\omega_n t}dt$$

Fourier coefficient a_n within a small range of frequency has a strength $|a_n|^2$, and its average is defined by the spectrum density or the power stpectrum. Here, the Wiener-Khinchin theorem shows that the power spectrum $I(\omega)$ is expressed by autocorrelation function $\phi(t)$ as follows:

$$I(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(t)e^{-i\omega t}dt.$$

The coefficient $\frac{1}{2\pi}$ is dual to the Fourier inverse transformation. According to the Wiener-Khinchin theorem, any stationary process can be obserbed by autocorrelation, and has the power spectrum. If the process has a specific distribution, the power spectrum has a simple form and becomes easy to analyse. For example, the power spectrum is constant over every frequency, the random effect has a white spectrum or is called by white noise.

Though the stationary process can apply only limited cases, as I explan next, there are some cases such as the Wiener process that has a stationary increment, which can be analysed by transforming the original process.

The next example is not a stationary process. For the time t and its interval from 0 to T, n points are chosen and labled sequentially.

$$0 = t_0 < t_1 < t_2 < \cdots < t_n < T$$

As time passes, the productivity y (for example the labour productivity and its training cost) is also fluctuating. As a rapid fluctuation (increases of labour productivity) costs more, the firm minimises its squared sum over n stages. Let $y_0 = 0,\ \ y_{n+1} = 0$ for convenience. Increase of productivity is a stochastic variable, and the increments $y_{i+1} - y_i$ are mutually independent. We focus on the adjustment cost that is increased by rapid labour productivity changes, and assume that unit cost from productivity fluctuation per unit of time is constant over the training stages i.

$$\frac{{y_1}^2}{t_1} + \frac{(y_2 - y_1)^2}{t_2 - t_1} + \cdots + \frac{(y_n - y_{n-1})^2}{t_n - t_{n-1}} + \frac{{y_n}^2}{T - t_n}.$$

The distribution of productivity fluctuations is defined as follows:

$$P_n((y_1, t_1), (y_2, t_2), \ldots, (y_n, t_n))$$
$$= \left(\frac{2\pi T}{\alpha}\right)^{1/2} \prod_{i=1}^{n-1} \left(\frac{\alpha}{2\pi(t_i - t_{i-1})}\right)^{1/2} \exp\left(-\frac{\alpha}{2}\frac{(y_i - y_{i-1})^2}{(t_i - t_{i-1})}\right) \quad (3.5)$$

This satisfies with the Kolmogorov's four conditions. But this is not stationary because of the autocorrelation.

$$< Y(t_1)Y(t_2) >= \frac{1}{\alpha}\frac{t_1(T - t_2)}{T}$$

Therefore, we cannot construct the power spectrum.

3.1.2 Transition probability and the Chapman-Kolmogorov equation

As the stationary model is very limited, we introduce the Markov property into the probability density hierarchy function P_n. There are many models with Markov process, but few are known as non-Markov process. Non-Markov process is a process with memory, but the memory term can be eliminated by "contraction" or "projection" and eventually converted into Markov process (Suzuki [1994]). In Markov process, the next relations are established.

$$\begin{aligned} & P_{1|n-1}((y_n, t_n)|(y_1, t_1), (y_2, t_2), \cdots, (y_{n-1}, t_{n-1})) \\ & \quad = \quad P_{1|1}(y_n, t_n|y_{n-1}, t_{n-1}) \\ & \quad \equiv \quad P(y_n, t_n|y_{n-1}, t_{n-1}) \end{aligned} \tag{3.6}$$

$P_{1|n-1}$ is a conditional probability density hierarchy function with $n-1$ pairs of conditions, and $P(y_n, t_n|y_{n-1}, t_{n-1})$ is a transition probability density.

In Markov process, the probability density hierarchy function can be expressed by the transition probability and its initial value $P_1(y_1, t_1)$. For example, at time t_1 the point was at y_1, at time t_2 the point moved to y_2, and at time t_3, the point moved to y_3. In this case, the probability density hierarchy is expressed by the conditional probability density, that is, the multiple of the transition probability from y_1 to y_2, and the conditional probability of y_3 given y_1 and y_2 as a condition. In case of Markov process, the latter probability is the transition probability of y_3 given the condition y_2.

$$\begin{aligned} & P_3((y_1, t_1), (y_2, t_2), (y_3, t_3)) \\ & = P_2((y_1, t_1), (y_2, t_2))P_{1|2}(y_3, t_3|(y_1, t_1), (y_2, t_2)) \\ & = P(y_3, t_3|y_2, t_2)P(y_2, t_2|y_1, t_1)P_1(y_1, t_1) \end{aligned} \tag{3.7}$$

The second line is integrated by y_2,

$$P_2((y_1, t_1), (y_3, t_3)) = P_1(y_1, t_1) \int P(y_3, t_3|y_2, t_2)P(y_2, t_2|y_1, t_1)dy_2.$$

From the definition of the conditional probability, under the condition that the point was y_1 at time t_1, the probability of y_3 at time t_3 is given as follows:

$$P(y_3, t_3|y_1, t_1) = \int P(y_3, t_3|y_2, t_2)P(y_2, t_2|y_1, t_1)dy_2 \tag{3.8}$$

This is the Chapman-Kolmogorov equation. In Markov process, all probability density hierarchy functions are determined by the transition probability. If it satisifies with the next consistency condition, any Markov process is determined by the Chapman-Kolmogorov equation.

$$P_1(y_2, t_2) = \int P(y_2, t_2|y_1, t_1)P_1(y_1, t_1)dy_1 \tag{3.9}$$

3.1.3 The master equation

In this section, taking Taylar expansion of the transition probability, we will derive the master equation, which is the transition probability in differential form with respect to time, i.e. the differential form of Chapman-Kolmogorov equation.

The transition probability defined on partial defferential with respect to time is

$$\frac{\partial P(y_2, t_2|y_1, t_1)}{\partial t_2} = \lim_{\tau \to 0} \frac{P(y_2, t_2 + \tau|y_1, t_1) - P(y_2, t_2|y_1, t_1)}{\tau}. \tag{3.10}$$

We will take Taylar expansion of $P(y_2, t_2 + \tau|y_1, t_1)$ with respect to τ for evaluation of the above definiton. At the same time, the integration of probability over the range must be normalised to one. That is

$$\int P(y_2, t_2|y_1, t_1))dy_2 = 1. \tag{3.11}$$

An expansion with the normalisation condition is expressed as follows:

$$P(y_2, t+\tau|y_1, t) = P(y_2, t|y_1, t)+\tau W_t(y_2|y_1)-\tau a_0 P(y_2, t|y_1, t)+o(\tau), \tag{3.12}$$

where $W_t(y_2|y_1) = \frac{P(y_2, t+\tau|y_1, t)}{\tau}$, a_0 is obtained by the normalisation condition:

$$a_0(y_1) = \int W_t(y|y_1)dy. \tag{3.13}$$

And $P(y_2, t|y_1, t)$ becomes Dirac's delta function by definition.

$$P(y_2, t|y_1, t) = \delta(y_2 - y_1),$$

where $o(\tau)$ is a term satisfied with $\lim_{\tau \to 0} o(\tau)/\tau = 0$.

Equation (3.12) is applied to the integrand $P(y_3, t_3|y_2, t_2)$ of the Chapman-Kolmogorov equation (3.8). That is $t_2 = t,\;\; t_3 = t + \tau$ etc., we obtain

$$\begin{aligned}
P(y_3, t_3|y_1, t_1) &= \int \delta(y_3 - y_2)(1 - \tau a_0) P(y_2, t_2|y_1, t_1) dy_2 \\
&\quad + \tau \int W_{t_2}(y_3|y_2) P(y_2, t_2|y_1, t_1) dy_2 \\
P(y_3, t_2 + \tau|y_1, t_1) &= P(y_3, t_2|y_1, t_1) \\
&\quad + \tau \int W_{t_2}(y_3|y_2) P(y_2, t_2|y_1, t_1) dy_2 \\
&\quad - \tau \int W_{t_2}(y|y_3) P(y_3, t_2|y_1, t_1) dy.
\end{aligned}$$

As $t_2 \to t_3$, the last integral variable of the right hand side y is changed to y_2,

$$\frac{\partial P(y_3, t_3|y_1, t_1)}{\partial t_3} = \int \{W_{t_3}(y_3|y_2) P(y_2, t_3|y_1, t_1) - W_{t_3}(y_2|y_3) P(y_3, t_3|y_1, t_1)\} \, dy_2 \;. \tag{3.14}$$

This is the master equation. Simply it can be written as

$$\frac{\partial P(y, t)}{\partial t} = \int \left(W_t(y|y') P(y', t) - W_t(y'|y) P(y, t) \right) dy'. \tag{3.15}$$

3.1.4 The Fokker-Planck equation

As the master equation is a conposite equation of differential and integral, although the analysis of it is easier than of the Chapman-Kolmogorov equation, much easier equation is preferable. In order to analyse more conveniently, we assume taking the short time enough not to allow a jump of y, but at the same time the equation keeps Markov property. This assumption derives the Fokker-Planck equation, which is a differential equation.[3]

Now using ξ of y's jump and inserting $y' = y - \xi$, the master equation becomes as follows:

$$\frac{\partial P(y, t)}{\partial t} = \int \left(W_t(y|y - \xi) P(y - \xi, t) - W_t(y - \xi|y) P(y, t) \right) d\xi.$$

Then we take Taylor expansion of $W_t(y|y - \xi) P(y - \xi, t)$ at $y = y - \xi$

[3] Fokker [1913] used a specail form, Planck [1917] derived it from the master equation, and Kolmogorov [1931] gave the mathematical foundation.

and $y' = y$ with respect to ξ.

$$\begin{aligned} W_t(y|y-\xi)P(y-\xi,t) &= W_t(y-\xi|y) - \frac{\partial}{\partial y'}\left(W_t(y-\xi|y)P(y,t)\right)\xi \\ &\quad + \frac{\partial^2}{\partial y'^2}\left(W_t(y-\xi|y)P(y,t)\right)\xi^2 + o(\xi^2). \end{aligned}$$

After inserting this equation into the original master equation, and exchange the differential operation under the integration, we obtain the following expansion:

$$\begin{aligned} \frac{\partial P(y,t)}{\partial t} &= -\frac{\partial}{\partial y'}\left(\int W_t(y-\xi|y)\xi d\xi P(y,t)\right) \\ &\quad +\frac{1}{2}\frac{\partial^2}{\partial y'^2}\left(\int W_t(y-\xi|y)\xi^2 d\xi P(y,t)\right) + \ldots. \end{aligned}$$

Here we assume that there exists $\delta > 0$ satisfied with the next three conditions.

$$\begin{array}{ll} W(y|y') \approx 0, \quad y' = y - \xi, & |\xi| > \delta \\ W(y|y' + \Delta y) \approx W(y|y'), & |\Delta y| < \delta \\ P(y + \Delta y, t) \approx P(y,t), & |\Delta y| < \delta \end{array}$$

Hence ignoring the higher order terms, the above expansion can be written as follows:

$$\begin{aligned} \frac{\partial P(y,t)}{\partial t} &= -\frac{\partial}{\partial y}\left(\int W_t(y-\xi|y)\xi d\xi P(y,t)\right) \\ &\quad +\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\int W_t(y-\xi|y)\xi^2 d\xi P(y,t)\right). \end{aligned}$$

The Fokker-Planck equation is obtained using the jump moment a_n, that is,

$$\begin{aligned} \frac{\partial P(y,t)}{\partial t} &= -\frac{\partial}{\partial y}\left(a_1(y)P(y,t)\right) \\ &\quad +\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(a_2(y)P(y,t)\right), \end{aligned} \tag{3.16}$$

$$\text{where} \quad a_n(y) = \int \xi^n W(y-\xi|y) d\xi.$$

If we take into account of all the higher terms of Taylor expansion, the equaiton becomes the Kramers-Moyal expansion. The Fokker-Planck equation is obtained under the condition of infinitesimal jump, but it is easy to analyse because it is a differential equation.[4].

The stochastic differential equation in financial economics is also a part of the examples of the Fokker-Planck equation. Therefore it requires the

[4]Section 3.2 shows solutions of some special cases, but only a few strict solustions have ever known.

same assumptions as the above. Non-Markov process can be approaximated by projection as a Markov process. But the infinitesimal jump is very restrictive, in addtion to Ornstein-Uhenbeck process that is stationary, Markovian, and Gaussian but does not include infinitesimal jump, the jump with a distribution of the quadratic funtion has been introduced.[5]

Here we assume that the total factor productivity obeys the stochastic process with infinitesimal jump. This assumption leads to the idea that the techinical progress is actually gradual. As we see the new technology that transcends many conditions of the time has long been forgotten from the society, a new technology may require some continuity to the previous when it spreads as an technical progress in the economy.

The Fokker-Planck equation is linear with respect to P. When we call a linear Fokker-Planck equation, we assume that $a_1(y)$ is a linear function of y, and $a_2(y)$ is constant. Conveniently, the Fokker-Planck equation can be determined and solvable, when the jump moments up to the second order, $a_1(y)$(the drift term) and $a_2(y)$(the diffusion term) are observable. Furthermore, these values are independent from time, you do not need to know all the history of the moments. In addition to such merits, the Fokker-Planck equation is related to the Langevin equation, which is easy to introduce random factors. However, there is a subtle problem in interpretation of the Langevin equation. In the linear case, there is no discrepancy between the data from macroscopic observation and the Fokker-Planck equation, but in the non-linear case, you cannot ignore effects from the diffusion coefficient $a_2(y)$, and it becomes difficult to identify $a_1(y)$. In such a case, the Ω expasion method for the master equation should be apllied.[6] If the equilibrium is unstable, the Ω expansion method for the master equation is not useful, then you should apply the scaling theory on order formation.[7].

3.1.5 The Langevin equation

We explain the Langevin equation that treats a random factor and that is widely used in physics. If the random factor distributes with Gaussian distribution, the Langevin equation is identical to the Fokker-Planck equaiton. Ito's stochastic differential equation are extremely often used in economics, but the random factor (diffution parameter) in the Langevin equation can be equally applicable with both Itp's interpretaion and Stratonovich's. But the different interpretation leads to the different Fokker-Planck equation.

[5]Babbs and Webber[1997] and El-Jahel et al.[1997] demonstrate that the distribution of an asset price fluctuation become leptokurtic without the infinitesimal jump.

[6]Ω is typically a parameter of the size of system.

[7]van Kampen[1992], p. 96, Suzuki[1994], p. 243.

Interpretation of the model depends on the system. Therefore careless interpretaion of the model leads a wrong solution for the system.

The Langevin equation is especially effective, when the macroscopic (deterministic) equation of the system is known and you would like to describe the effect of fluctuation (Langevin force, $L(t)$). For example by Frisch [1933], the macro economic system can be described as a motion equation of a harmonic oscillator. Extrenal factor outside from the economy, for example a random factor such as climate change $L(t)$ affects the economy. In this case, the logarithm of the price level p and a random Langevin force $L(t)$ are connected by the following equation.

$$\frac{d^2p}{dt^2} + \gamma\frac{dp}{dt} + \omega^2 p = L(t). \tag{3.17}$$

We assume further on $L(t)$,

$$\begin{aligned} < L(t) > &= 0 \\ < L(t)L(t') > &= \Gamma\delta(t-t'). \end{aligned} \tag{3.18}$$

It is important to note that the stochastic property of $L(t)$ is independent from p, and that shock of $L(t)$ is approximated by δ function. The shock continues infinitesimal interval, and its strength is infinite, but this is an approximation. If the shock has finite interval, Ito's interpretation becomes inappropriate.

To solve the Langevin equation (3.17), first we solve the homogeneous equation, and apply the constant variation method to it and obtain the general solution. Let the solutions of the characteristic equation are μ_1 and μ_2, then the general solution is expressed by the following equations.

$$p(t) = c_1(t)e^{\mu_1 t} + c_2(t)e^{\mu_2 t}, \quad \mu_i = -\gamma/2 \pm \sqrt{\gamma^2/4 - \omega^2}, \quad i = 1, 2.$$

For the degenerate case, it is a well known solution as follows:

$$p(t) = (c_1(t) + c_2(t)t)te^{-\gamma t/2}.$$

When we use the constant variation method, we can choose an appropriate condition between $c_1(t)$ and $c_2(t)$. If we choose

$$\frac{dc_1}{dt}e^{\mu_1 t} + \frac{dc_2}{dt}e^{\mu_2 t} = 0,$$

and integrate for the equation systems of dc_1/dt and dc_2/dt. Then we obtain the general solustion for $p(t)$. If we denote c_{10} and c_{20} as integral constants,

$$\begin{aligned} p(t) &= \tfrac{e^{\mu_1 t}}{\mu_1-\mu_2}\int_0^t L(s)e^{-\mu_1 s}ds + \tfrac{e^{\mu_2 t}}{\mu_2-\mu_1}\int_0^t L(s)e^{-\mu_2 s}ds + c_{10}e^{\mu_1 t} \\ &\quad + c_{20}e^{\mu_2 t}. \end{aligned}$$

If we use The initial condition $p(0) = p_0, \; dp(0)/dt = \dot{p}_0$, the solution becomes

$$\begin{aligned} p(t) &= (1/D)\dot{p}_0 e^{-\gamma t/2}\sinh(Dt) \\ &\quad + (1/2)p_0 e^{-\gamma t/2}\{\cosh(Dt) + \gamma/2\sinh(Dt)\} \\ &\quad + \int_0^t e^{-\gamma(t-s)/2}\{\cosh(D(t-s)) \\ &\quad - \gamma/(2D)\sinh(D(t-s))L(s)ds\}. \end{aligned}$$

Where $D = \sqrt{\gamma^2/4 - \omega^2}$. $p(t)$'s mean $< p(t) >$ is calculated by

$$\begin{aligned} < p(t) > &= (1/D)\dot{p}_0 e^{-\gamma t/2}\sinh(Dt) \\ &\quad + (1/2)p_0 e^{-\gamma t/2}\{\cosh(Dt) + \gamma/2\sinh(Dt)\}. \end{aligned}$$

And the second moment $< p(t)^2 >$ is as follows:

$$\begin{aligned} < p(t)^2 > &= \{< p(t) >\}^2 + \tfrac{\Gamma}{2\omega^2\gamma} \\ &\quad - \tfrac{\Gamma e^{\gamma t}}{4D^2}\left\{\tfrac{2}{\gamma} - \tfrac{1}{\omega^2}\left(\tfrac{\gamma}{2}\cosh(2Dt) - \sinh(2Dt)\right)\right\}. \end{aligned}$$

The variance $< p(t)^2 > -\{< p(t) >\}^2$ becomes $\Gamma/(2\omega^2\gamma)$ as $t \to \infty$. Economically this value means a variance of the price in equilibrium, there is no further meanings. We cannot know the equilibrium can be realised as time goes infinity. Even if the equilibrium attained in the long run, there is no relation between the variance of the prices and the other macroeconomic variables.

But in physics, the statistical mechanics in equilibrium shows this value is equvalent to kT/ω^2 from the equal energy distribution law.[8] As a result, the equation $\Gamma = \gamma kT$, that is, the relation between a micro parameter and a macro variable temperature T is obtained.

It is this equation that Einstein [1905] proves the fluctuation-dissipation theorem in the simplest form, which relates between fluctuation Γ and dissipation γ.[9] The Langevin's approach is extremely efficient and widely used,

[8] k is the Boltzman constant, T is the absolute temperature, in (3.17) mass of the harmonic oscillator is normalised to one.

[9] Einstein derives it from the thermodynamical identities and the balance equation between the number of particles that pass per unit of time and per area in equilibrium.

because it can derive the fluctuation-dissipation theorem in general. Unfortunately, there is no relation such as Einstein's relation in economics, therefore it lacks the core of these types of analyses.

Up to now, we did not give any distribution to the Langevin term $L(t)$, we use only the second and the first moments. As a result, we cannot determine the distribution of $p(t)$. But the Fokker-Planck equation determines the shape of distribution, becase it is a partial differential equation. Then if the shape of a distribution is determined by the first and the second moments of $L(t)$ $(Gauss)$, in other words, we assume the Langevin equation that $W(t) = \int^t L(s)ds$ and $W(t)$ obeys a Wiener process, there is a possibility of equivalence between the Langevin eqution and the Fokker-Planck equation.

I will show first that the Langevin equation becomes the Fokker-Planck equation without any problem, according to van Kampen [1992]. And next I will explain another example by van Kampen [1992] including some difficulties.

The former case is the fisrt order differential equation with non-linear dissipation term, that is

$$dx/dt = A(x) + L(t).$$

Given the initial value $x(0)$ and a sample function of $L(t)$, $x(t)$ is determnied. $L(t)$ is statistically independent at different time points, $x(t)$ is a Markov process. Since it is governed by the master equation, it can be expand into the Kramers-Moyal expansion. The third and higher order terms diminish at the limit $\Delta t \to 0$, the second and lower order coefficients are comparable with the coefficients of the Fokker-Planck equation. Denote $\Delta x(t) = x(t + \Delta t) - x(t)$,

$$\Delta x(t) = \int_t^{t+\Delta t} A(x(s))ds + \int_t^{t+\Delta t} L(s)ds.$$

Taking average of this equation,

$$< \Delta x(t) >= A(x(t))\Delta t + O(\Delta t)^2.$$

And the second order moment is

$$
\begin{aligned}
< \Delta x(t)^2 > \quad &= \quad < \left\{ \int_t^{t+\Delta t} A(x(s))ds \right\}^2 > \\
&+2\int_t^{t+\Delta t} ds \int_t^{t+\Delta t} ds' < A(x(s))L(s') > \\
&+\int_t^{t+\Delta t} ds \int_t^{t+\Delta t} ds' < L(s)L(s') > .
\end{aligned}
$$

The first term of the right hand side diminishes as Δt^2 and the second term is

$$A(x(t+\Delta t)) = A(x(t)) + A'(x(t))(x(t+\Delta t) - x(t))\Delta t + \cdots .$$

As a result,

$$
\begin{aligned}
&2A(x(t))\Delta t \textstyle\int_t^{t+\Delta t} ds < L(s) > \\
&+2A'(x(t)) \textstyle\int_t^{t+\Delta t} ds \int_t^{t+\Delta t} ds < (x(s) - x(t))L(s') > ds' \\
&+\ldots .
\end{aligned}
$$

The first term disappears, and the second term is $o(\Delta t)$, and the higher order terms are $o(\Delta t)$ at most. The last term is $\Gamma\Delta t$. Then the first order moment is $A(x(t))$, the second order moment is Γ, and the equation becomes the Fokker-Planck equation as follows:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial (A(x)P(x,t))}{\partial x} + \frac{\Gamma}{2}\frac{\partial^2 P(x,t)}{\partial x^2}. \tag{3.19}$$

Next we consider the Langevin equation as follows:

$$dx/dt = A(x) + C(x)L(t). \tag{3.20}$$

In this case, as we assume that $C(x) \neq 0$, and divide both side of the equation by it,

$$1/C(x)dx/dt = A(x)/C(x) + L(t).$$

If we denote $X = \int^X 1/C(x)dx$ and $\bar{A}(X) = A(x)/C(x)$, and rearranged the equation as

$$dX/dt = \bar{A}(X) + L(t).$$

This equation can be expressed by analogy,

$$\frac{\partial \bar{P}(X,t)}{\partial t} = -\frac{\partial (\bar{A}(X)\bar{P}(X,t))}{\partial X} + \frac{\Gamma}{2}\frac{\partial^2 \bar{P}(X,t)}{\partial X^2}.$$

Let $\bar{P}(X,t) = P(x,t)C(x)$.[10] In order to convert it into the original variable x, we insert

$$\frac{\partial \bar{P}}{\partial X} = \frac{\partial x}{\partial X}\left(P\frac{\partial C}{\partial x} + C\frac{\partial P}{\partial x}\right)$$

and

$$\begin{aligned}\frac{\partial^2 \bar{P}}{\partial X^2} &= C(x)\left(C'(x)^2 + C(x)C''(x)\right)P(x,t) + C(x)^2 3C'(x)\frac{\partial P(x,t)}{\partial x} \\ &+C(x)^3\frac{\partial^2 P(x,t)}{\partial x^2}\end{aligned}$$

into the above equation. Using the relation $\frac{\partial x}{\partial X} = C(x)$,

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial y}\left(A(x)P(x,t)\right) + \frac{\Gamma}{2}\frac{\partial}{\partial x}\left\{C(x)\frac{\partial (C(x)P(x,t))}{\partial x}\right\}. \quad (3.21)$$

Sorting by $P(x,t)$ gives the equation

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial y}\left\{\left(A(x) + \frac{\Gamma}{2}C(x)C'(x)\right)P(x,t)\right\} + \frac{\Gamma}{2}\frac{\partial^2 (C(x)^2 P(x,t))}{\partial x^2}. \quad (3.22)$$

This Fokker-Planck equation is not a simple transformation of the Langevin equation (3.18). Applying the same procedure as we derived the Fokker-Planck equation (3.19), (3.18) is integreted over the interval $(t, t+\Delta t)$. And we interpret the second term of the righ hand side as follows:

$$x(t+\Delta t) - x(t) = \int_t^{t+\Delta t} A(x(s))ds + C\left(\frac{x(t+\Delta t)+x(t)}{2}\right)\int_t^{t+\Delta t} L(s)ds. \quad (3.23)$$

This give the first moment

$$< x(t+\Delta t) - x(t) >= A(x(t))\Delta t + \langle C\left(\frac{x(t+\Delta t)+x(t)}{2}\right)\int_t^{t+\Delta t} L(s)ds\rangle.$$

The second term is expanded by $C((x(t+\Delta t) + x(t))/2) = C(x(t)) + C'(x(t))(x(t+\Delta t) - x(t))/2 + o(\Delta t)$, using $x(t+\Delta t) - x(t)$ into (3.23)

[10] The first order moment of the original equation is recovered by the condition:

$$\frac{\partial \bar{P}}{\partial P} = \frac{\bar{P}}{P}.$$

The integral constant is $C(x)$ that is the most simplest function of x and consistent with the second order moment.

and the next relation:

$$
\begin{aligned}
&\langle C\left(\tfrac{x(t+\Delta t)+x(t)}{2}\right)\int_t^{t+\Delta t} L(s)ds\rangle \\
&= \tfrac{C'(x(t))}{2}\langle (x(t+\Delta t)-x(t))\int_t^{t+\Delta t} L(s)ds\rangle \\
&= \tfrac{C'(x(t))}{2}\langle\, (A(x(t))\Delta t \\
&\quad +C\left(\tfrac{x(t+\Delta t)+x(t)}{2}\right)\int_t^{t+\Delta t} L(s)ds\Big)\int_t^{t+\Delta t} L(s)ds\rangle \\
&= \tfrac{C'(x(t))}{2}\langle\, \Big(C(x(t))+C'(x(t))\tfrac{x(t+\Delta t)-x(t)}{2}+o(\Delta t)\Big) \\
&\quad \times\int_t^{t+\Delta t} L(s)ds\int_t^{t+\Delta t} L(s)ds\rangle \\
&= \tfrac{C'(x(t))C(x(t))\Gamma}{2}\Delta t+o(\Delta t).
\end{aligned}
$$

Therefore,

$$
< x(t+\Delta t)-x(t) >= A(x(t))\Delta t+\frac{C'(x(t))C(x(t))\Gamma}{2}\Delta t+o(\Delta t).
$$

The second moment can be attained by the similar procedure as

$$
\langle\, (x(t+\Delta t)-x(t))^2\,\rangle = C(x(t))^2\Gamma\Delta t+o(\Delta t).
$$

From these two moment, the corresponding Fokker-Planck equation is the equation (3.22). The interpretation that the Langevin equation (3.18) leads to (3.23) is by Stratonovich. On the other hand, Ito's interpretaion is

$$
x(t+\Delta t)-x(t)=\int_t^{t+\Delta t} A(x(s))ds+C(x(t))\int_t^{t+\Delta t} L(s)ds. \quad (3.24)
$$

In this case, the corresponding Fokker-Planck equation is

$$
\frac{\partial P(x,t)}{\partial t}=-\frac{\partial}{\partial y}\left(A(x)P(x,t)\right)+\frac{\Gamma}{2}\frac{\partial^2(C(x)^2P(x,t))}{\partial x^2}. \quad (3.25)
$$

But in this situation, the rules of variables conversion for the Langevin equation must be changed. These rules are, as well known, the Ito's lemma. The problem is that the solustion and the Fokker-Planck equation are changed by the interpretaion of the Langevin equation. It depneds on whether we use the diffusion coefficient $C(x)$ evaluated just before the jump or evaluated at the average between before and after the jump, for the term $C(x)L(t)$ in the non-linear Langevin equation. Using the diffusion coefficient just before the jump by Ito's interpretation, this term becomes martingale, it is mathematically convenient because all results for the martingale can be applicable. However the physical phenomena do not necessarily follow the

Ito's assumption, and the model often becomes for the interpretation of Stratonovich.

For example, a Brownian motion particle jumps inelastic, it is an extreme abstraction that the jump occurs at the moment of colusion. But as Einstein considered, the diffusion coefficient of the Brownian motion is determined by the absolute temperature, the viscosity of fluid and the size of the particle. If C is depend on x and has a gradient, the experiment can cotrol constancy of the diffusion coefficient. The problem arises when the non-linearity is significant and the fluctulation is difficult to identify the drift and the diffusion separately. If one can cut the source of the noise the Langevin term $L(t)$, the system can be identifiable. It is the same for the economic model, but it is especially difficult to cut the source of the noise $L(t)$. It is also very difficult to control by the experiment in order to eliminate the gradient of the diffution coefficient. Identifying the source of the noise, we must decide which interpretaion is appropriate, when we use the non-linear Langevin equation in economic model (Mori [1975]).

3.2 Solutions to the Fokker-Planck equation

If the Langevin equation and its interpretation are formulated, the remaining is to solve the Fokker-Planck equation. It is true that the Fokker-Planck equation is easier to use than the master equation, becase it is a partial differential equation. Actually only a few of the strict solutions are known.[11] This is the same to the solutions of the corresponding Langevin equation.

But there are many contributions to the method of solving the Fokker-Planck equation. For example, Bluman [1971], Bluman and Kumei [1989], Cukier et al. [1973], Dresner [1983], Feller [1950], Hill [1982], Nariboli [1977]. Among them, the most general and systematic method is an application of one parameter Lie group.[12].

I will explain the idea breafly as follows: first derive the invariant surface and its Lie group that keeps the partial differential equation invariant,

[11] Rogers [1997] describes Gaussisan Brown motion, O-U process, and Bessel process and said that almost all solutions are listed by these three. van Kampen [1992] explains the same situation.

[12] Bluman [1971], Bluman and Kumei [1989], Hill [1982], Nariboli [1977] use this method. Steinberg [1977] explains more general partial differential equation. Bluman [1971] explains the constant diffusion term. Nariboli [1977] systematically classifys the Fokker-Planck equation and presents many solutions. Hill [1982] expand the contributions of Nariboli. Bluman and Kumei [1989] generalise the method to the multi-parameter Lie group, and show the solution of the ordinal differential equation. Calculation to find the solution is computationally intensive, several algorithms have been developed (Zwillinger [1997]).

second reduce the independent variables of the partial differential equation using the invariant surface, third solve the ordinary differential equation, and finally recover the transformed valiables into the original space. There is a similarity method, but this method can find all possible similarity variables. Even if the transformation cannot derive separation of variables, the approximation can be efficient using a transformation close to the invariant surface (Dresner[1983], Chapter 7).

It is difficult to explain all about the method here, and I recommend to refer the proper books on partial differential equations. Here I conentrate to the Fokker-Planck equation (parabolic partial differential equation). The above equations (3.22) and (3.25) are generally transformed into the following equations:

$$a(x)\frac{\partial^2 p(x,t)}{\partial x^2} + b(x)\frac{\partial p(x,t)}{\partial x} + c(x)p(x,t) = \frac{\partial p(x,t)}{\partial t}. \tag{3.26}$$

To construct the Lie group, the following transformation is introduced by an infinitesimal parameter ϵ,

$$\begin{aligned} \bar{x} &= x + \epsilon X(p,x,t) + O(\epsilon^2) \\ \bar{t} &= t + \epsilon T(p,x,t) + O(\epsilon^2) \\ \bar{p} &= p + \epsilon P(p,x,t) + O(\epsilon^2). \end{aligned} \tag{3.27}$$

Where $O(\epsilon^2)$ is a small number of the same order as ϵ^2.[13] Calculating the Jacobian, there is a following relation between the independent variables x, t, $\bar{x}$, and $\bar{t}$:

$$\begin{aligned} \frac{\partial x}{\partial \bar{x}} &= 1 - \epsilon\left(\frac{\partial X}{\partial x} + \frac{\partial X}{\partial p}\frac{\partial p}{\partial x}\right) + O(\epsilon^2) \\ \frac{\partial x}{\partial \bar{t}} &= -\epsilon\left(\frac{\partial X}{\partial t} + \frac{\partial X}{\partial p}\frac{\partial p}{\partial t}\right) + O(\epsilon^2) \\ \frac{\partial t}{\partial \bar{x}} &= -\epsilon\left(\frac{\partial T}{\partial x} + \frac{\partial T}{\partial p}\frac{\partial p}{\partial x}\right) + O(\epsilon^2) && (3.28) \\ \frac{\partial t}{\partial \bar{t}} &= 1 - \epsilon\left(\frac{\partial T}{\partial t} + \frac{\partial T}{\partial p}\frac{\partial p}{\partial t}\right) + O(\epsilon^2). && (3.29) \end{aligned}$$

[13] We say that

$$f(x) = O(x) \text{ as } x \to x_0$$

if there exists a positive constant C and a neighborhood of U of x_0 such that

$$|f(x)| \le C|x| \text{ for all } x \text{ in } U$$

.

Using this relation, we can obtain all the differential rules appeared in the transformed equation.

$$\begin{aligned}\frac{\partial \bar{p}}{\partial \bar{x}} &= \frac{\partial \bar{p}}{\partial x}\frac{\partial x}{\partial \bar{x}} + \frac{\partial \bar{p}}{\partial t}\frac{\partial t}{\partial \bar{x}} \\ \frac{\partial \bar{p}}{\partial \bar{t}} &= \frac{\partial \bar{p}}{\partial x}\frac{\partial x}{\partial \bar{t}} + \frac{\partial \bar{p}}{\partial t}\frac{\partial t}{\partial \bar{t}}\end{aligned}$$

Evaluating these equations in terms of ϵ's order, we obtain as follows:

$$\begin{aligned}\frac{\partial \bar{p}}{\partial \bar{x}} &= \frac{\partial p}{\partial x} + \epsilon\left\{\frac{\partial P}{\partial x} - \frac{\partial X}{\partial x}\frac{\partial p}{\partial x} - \frac{\partial T}{\partial x}\frac{\partial p}{\partial t}\right. \\ &\quad \left. + \left(\frac{\partial P}{\partial p} - \frac{\partial X}{\partial p}\frac{\partial p}{\partial x} - \frac{\partial T}{\partial p}\frac{\partial p}{\partial t}\right)\frac{\partial p}{\partial x}\right\} + O(\epsilon^2) \\ \frac{\partial \bar{p}}{\partial \bar{t}} &= \frac{\partial p}{\partial t} + \epsilon\left\{\frac{\partial P}{\partial t} - \frac{\partial X}{\partial t}\frac{\partial p}{\partial x} - \frac{\partial T}{\partial t}\frac{\partial p}{\partial t}\right. \\ &\quad \left. + \left(\frac{\partial P}{\partial p} - \frac{\partial X}{\partial p}\frac{\partial p}{\partial x} - \frac{\partial T}{\partial p}\frac{\partial p}{\partial t}\right)\frac{\partial p}{\partial t}\right\} + O(\epsilon^2).\end{aligned}$$

The second order differential rules can be obtained as the same procedure but with more complicated, using the following expressions, substitutions and operations.

$$\begin{aligned}\frac{\partial^2 \bar{p}}{\partial \bar{x}^2} &= \frac{\partial x}{\partial \bar{x}}\frac{\partial}{\partial x}\left(\frac{\partial \bar{p}}{\partial \bar{x}}\right) + \frac{\partial t}{\partial \bar{x}}\frac{\partial}{\partial t}\left(\frac{\partial \bar{p}}{\partial \bar{x}}\right) \\ \frac{\partial^2 \bar{p}}{\partial \bar{x}\partial \bar{t}} &= \frac{\partial x}{\partial \bar{t}}\frac{\partial}{\partial x}\left(\frac{\partial \bar{p}}{\partial \bar{x}}\right) + \frac{\partial t}{\partial \bar{t}}\frac{\partial}{\partial t}\left(\frac{\partial \bar{p}}{\partial \bar{x}}\right) \\ \frac{\partial^2 \bar{p}}{\partial \bar{t}^2} &= \frac{\partial x}{\partial \bar{t}}\frac{\partial}{\partial x}\left(\frac{\partial \bar{p}}{\partial \bar{t}}\right) + \frac{\partial t}{\partial \bar{t}}\frac{\partial}{\partial t}\left(\frac{\partial \bar{p}}{\partial \bar{t}}\right).\end{aligned}$$

Evaluating the first order of ϵ, the sencond order derivatives become as follwos:

$$\begin{aligned}\frac{\partial^2 \bar{p}}{\partial \bar{x}^2} &= \frac{\partial^2 p}{\partial x^2} + \epsilon\left\{\frac{\partial P}{\partial x^2} + \left(2\frac{\partial^2 P}{\partial p \partial x} - \frac{\partial^2 X}{\partial x^2}\right)\frac{\partial p}{\partial x} - \frac{\partial^2 T}{\partial x^2}\frac{\partial p}{\partial t}\right. \\ &\quad + \left(\frac{\partial^2 P}{\partial p^2} - 2\frac{\partial^2 X}{\partial p \partial x}\right)\left(\frac{\partial p}{\partial x}\right)^2 - 2\frac{\partial^2 T}{\partial p \partial x}\frac{\partial p}{\partial t}\frac{\partial p}{\partial x} - \frac{\partial^2 X}{\partial p^2}\left(\frac{\partial p}{\partial x}\right)^3 \\ &\quad - \frac{\partial^2 T}{\partial p^2}\frac{\partial p}{\partial t}\left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial p} - 2\frac{\partial X}{\partial x} - 3\frac{\partial X}{\partial p}\frac{\partial p}{\partial x} - \frac{\partial T}{\partial p}\frac{\partial p}{\partial t}\right)\frac{\partial^2 p}{\partial x^2} \\ &\quad \left. - 2\left(\frac{\partial T}{\partial x} + \frac{\partial T}{\partial p}\frac{\partial p}{\partial x}\right)\frac{\partial^2 p}{\partial x \partial t}\right\} + O(\epsilon^2).\end{aligned}$$

The next second order derivative does not appear in the Fokker-Planck equation, but I will write down for future reference.

$$\begin{aligned}\frac{\partial^2\bar{p}}{\partial\bar{x}\partial\bar{t}} &= \frac{\partial^2 p}{\partial x\partial x}+\epsilon\left\{\frac{\partial P}{\partial x\partial t}+\left(\frac{\partial^2 P}{\partial p\partial t}-\frac{\partial^2 X}{\partial x\partial t}\right)\frac{\partial p}{\partial x}\right.\\ &+\left(\frac{\partial^2 P}{\partial x\partial t}-\frac{\partial^2 T}{\partial x\partial t}\right)\frac{\partial p}{\partial t}-\frac{\partial^2 X}{\partial p\partial t}\left(\frac{\partial p}{\partial x}\right)^2\\ &+\left(\frac{\partial^2 P}{\partial p^2}-\frac{\partial^2 X}{\partial p\partial x}-\frac{\partial^2 T}{\partial p\partial t}\right)\frac{\partial p}{\partial x}\frac{\partial p}{\partial t}-\frac{\partial^2 T}{\partial p\partial x}\left(\frac{\partial p}{\partial t}\right)^2\\ &-\frac{\partial^2 X}{\partial p^2}\left(\frac{\partial p}{\partial t}\right)\left(\frac{\partial p}{\partial x}\right)^2-\frac{\partial^2 T}{\partial p^2}\left(\frac{\partial p}{\partial x}\right)\left(\frac{\partial p}{\partial t}\right)^2\\ &-\left(\frac{\partial X}{\partial t}+\frac{\partial X}{\partial p}\frac{\partial p}{\partial t}\right)\frac{\partial^2 p}{\partial x^2}\\ &+\left(\frac{\partial P}{\partial p}-\frac{\partial X}{\partial x}-\frac{\partial T}{\partial t}-2\frac{\partial X}{\partial p}\frac{\partial p}{\partial x}-2\frac{\partial T}{\partial p}\frac{\partial p}{\partial t}\right)\frac{\partial^2 p}{\partial x\partial t}\\ &\left.-\left(\frac{\partial T}{\partial x}+\frac{\partial T}{\partial p}\frac{\partial p}{\partial x}\right)\frac{\partial^2 p}{\partial t^2}\right\}+O(\epsilon^2).\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\bar{p}}{\partial\bar{t}^2} &= \frac{\partial^2 p}{\partial t^2}+\epsilon\left\{\frac{\partial P}{\partial t^2}+\left(2\frac{\partial^2 P}{\partial p\partial t}-\frac{\partial^2 T}{\partial t^2}\right)\frac{\partial p}{\partial t}-\frac{\partial^2 X}{\partial t^2}\frac{\partial p}{\partial x}\right.\\ &+\left(\frac{\partial^2 P}{\partial p^2}-2\frac{\partial^2 T}{\partial p\partial t}\right)\left(\frac{\partial p}{\partial t}\right)^2-2\frac{\partial^2 X}{\partial p\partial t}\frac{\partial p}{\partial x}\frac{\partial p}{\partial t}-\frac{\partial^2 T}{\partial p^2}\left(\frac{\partial p}{\partial t}\right)^3\\ &-\frac{\partial^2 X}{\partial p^2}\frac{\partial p}{\partial x}\left(\frac{\partial p}{\partial t}\right)^2+\left(\frac{\partial P}{\partial p}-2\frac{\partial T}{\partial t}-3\frac{\partial T}{\partial p}\frac{\partial p}{\partial t}-\frac{\partial X}{\partial p}\frac{\partial p}{\partial x}\right)\frac{\partial^2 p}{\partial t^2}\\ &\left.-2\left(\frac{\partial X}{\partial t}+\frac{\partial X}{\partial p}\frac{\partial p}{\partial t}\right)\frac{\partial^2 p}{\partial t\partial x}\right\}+O(\epsilon^2)\end{aligned}$$

To derive the invariant surface using these transformation, due to the invariance of the solution p,

$$\bar{p}(p,x,t;\epsilon)=p(\bar{x},\bar{t}).$$

Expanding the equation of the first order of ϵ,

$$P(p,x,t)=\frac{\partial p}{\partial x}X(p,x,t)+\frac{\partial p}{\partial t}T(p,x,t).$$

Solutions of this equation are the similarity variables that reduce the independent variable appeared in the equation. The characteristic equation is

$$\frac{dP}{P} = \frac{dX}{X} = \frac{dT}{T}. \tag{3.30}$$

Next, according to the differential rules, derive the relation between the variables that keep the Fokker-Planck equation invariant, that is

$$\frac{\partial p(x,t)}{\partial t} - a(x)\frac{\partial^2 p(x,t)}{\partial x^2} - b(x)\frac{\partial p(x,t)}{\partial x} - c(x)p(x,t)$$
$$= \frac{\partial \bar{p}(\bar{x},\bar{t})}{\partial \bar{t}} - a(\bar{x})\frac{\partial^2 \bar{p}(\bar{x},\bar{t})}{\partial \bar{x}^2} - b(\bar{x})\frac{\partial \bar{p}(\bar{x},\bar{t})}{\partial \bar{x}} - c(\bar{x})\bar{p}(\bar{x},\bar{t}).$$

The following equations are used and substituted in the differential operator, $\bar{x}$, $\bar{t}$, $\bar{p}$ the difinition (3.28), and the coefficients

$$a(\bar{x}) = a(x) + a'(x)\epsilon X, \quad b(\bar{x}) = b(x) + b'(x)\epsilon X, \quad c(\bar{x}) = c(x) + c'(x)\epsilon X.$$

$'$ means d/dx the derivative with respect to x. The coefficients of ϵ is equal to zero, therefore we can rearrange the partial differential coefficient further. I have omitted the calculations, it is too long to describe, in order to establishe the identity, every coefficient of each partial derivative must be zero.

I can obtain the following conditions from the coefficients of $(\partial p/\partial t)^2$, $(\partial^2 p/\partial x^2)(\partial p/\partial x)$, $(\partial^2 p/\partial t \partial x)$, $(\partial p/\partial x)^2$:

$$\begin{aligned}
\tfrac{\partial X}{\partial p} = 0 \mathit{derives} \quad & X = X(x,t) \\
\tfrac{\partial T}{\partial p} = 0,\ \tfrac{\partial T}{\partial x} = 0 \mathit{derive} \quad & T = T(t) \\
\tfrac{\partial^2 P}{\partial p^2} = 0 \mathit{derives} \quad & P = p(x,t)f(x,t) + g(x,t).
\end{aligned}$$

The coefficient of $\partial^2 p/\partial x^2$ derives

$$-a'(x)X(x,t) - a(x)f(x,t) + 2a(x)\frac{\partial X(x,t)}{\partial x} = 0.$$

Integrating this gives $X(x,t)$:

$$X(x,t) = X_0(t)\sqrt{a} + \frac{1}{2}\frac{dT}{dt}\sqrt{a}\int a^{-1/2}dx.$$

X_0 is independent from x, and the integral constant. The coefficient of $\partial p/\partial x$ derives the next equation:

$$2\frac{\partial f}{\partial x} = \frac{\partial^2 X}{\partial x^2} + \frac{b}{a}\left(\frac{\partial X}{\partial x} - \frac{dT}{dt}\right) - \frac{X}{a}b'(x) - \frac{1}{a}\frac{\partial X}{\partial t}.$$

This equation is also integrable, and we can obtain $f(x,t)$,

$$\begin{aligned} f \;&= f_0(t) + \tfrac{X_0(t)}{2}\left(\tfrac{da^{1/2}}{dx} - ba^{-1/2}\right) \\ &+\tfrac{T'(t)}{4}\left(\tfrac{d\left(I(x)a^{1/2}\right)}{dx} - I(x)a^{-1/2}b\right) - \tfrac{I(x){X_0}'(t)}{2} - \tfrac{I^2(x)T''(t)}{8}, \end{aligned} \tag{3.31}$$

where $I(x) = \int a^{-1/2}dx$.

Finally, the next equation can be obtained

$$\begin{aligned} &\frac{\partial g}{\partial t} - cg - b\frac{\partial g}{\partial x} - a\frac{\partial^2 g}{\partial x^2} = 0 \\ &\frac{\partial f}{\partial t} - cT'(t) - b\frac{\partial f}{\partial x} - a\frac{\partial^2 f}{\partial x^2} - X\frac{dc}{dx} = 0. \end{aligned}$$

The partial differential equation of g is the same as the p's Fokker-Planck equation, hence $g = 0$ in general. The equation of f is substituted by the previous equation (3.31). And rearranging further, it can be shown the next equation.

$$\begin{aligned} &\frac{df_0}{dt} + \frac{T''(t)}{2} - \frac{I(x){X_0}''(t)}{2} - \frac{I^2(x)T'''(x)}{8} \\ &\quad = \frac{X_0}{2}\left\{\left(a\frac{d^2}{dx^2} + b\frac{d}{dx}\right)\left(\frac{da^{1/2}}{dx} - a^{-1/2}b\right) + 2a^{1/2}\frac{dc}{dx}\right\} \\ &\quad +\frac{T'(t)}{4}\left\{\left(a\frac{d^2}{dx^2} + b\frac{d}{dx}\right)\left(\frac{(da^{1/2}I(x))}{dx} - a^{-1/2}bI(x)\right)\right. \\ &\quad \left. +4c + 2I(x)a^{1/2}\frac{dc}{dx}\right\}. \end{aligned} \tag{3.32}$$

Up to this, the relations hold for any Fokker-Planck equation. But next we have to substitute $a(x)$, $b(x)$, $c(x)$ for some special cases. and compare the coefficients of the power of x appeared in (3.32). As a result, the functional forms of f_0, X_0, T are derived. And the functional forms of the transformation group X, T, P are determined. I will show two of the most general examples by Bluman [1971] and Hill [1982].

$\frac{\partial^2 p(x,t)}{\partial x^2} + \frac{\partial b(x)p(x,t)}{\partial x} = \frac{\partial p(x,t)}{\partial t}$ [14]

The initial condition is assumed to be

$$p(x,0) = \delta(x - x_0), \quad x(0) = x_0 > 0.$$

[14] Bluman [1971] derives the solution, and Bluman and Kumei [1989] (pp. 226–232) explains in detail.

Substituting $a(x) = 1$ and $c(x) = b'(x)$ into (3.32) gives $I(x) = x$. The left hand side of the equation is a quadratic equation of x. After taking the derivatives three times, the equation becomes zero, and the right hand side of the equation gives the differential equation with respect to b.

$$2X_0(t)(b'' - b'b)''' + dT(t)/dt(xb'' + 2b' - bb'x - b^2)''' = 0.$$

Because $b(x)$ is the drift term, it is an even function $b(x) = -b(-x)$. Therefore when $dT(t)/dt \neq 0$, the next equation must be satisfied.

$$(xb'' + 2b' - bb'x - b^2)''' = 0$$

Integrating thie equation gives the next equation:

$$2b'(x) - b^2(x) - 4\beta^2 - \gamma + (16\nu^2 - 1)/x^2.$$

β, γ, ν are the integral constants, and chosen for the convenience of the following calculations. This is a Riccati type of the differential equation, and transformed by $b(x) = -2V'(x)/V(x)$ it becomes

$$\frac{d^2V}{dV^2} + \left(\frac{\gamma}{4} - \frac{x^2\beta^2}{4} - \frac{16\nu^2 - 1}{4x^2}\right)V = 0.$$

The solution of this ordinal differential equation is expressed by the Kummer's first type of confluent hypergeometric function $F(\lambda, \mu; z)$.

$$V(x) = \left(\frac{\beta x^2}{2}\right)^{1/4+\nu} e^{-\beta x^2/4} F(\lambda, \mu; \frac{\beta x^2}{2}),$$

where $\mu = 2\nu + 1$, $\lambda = \nu + 1/2 - \gamma/(8\beta)$.[15] This is the invariant equation under the infinitesimal transformation (3.28), and the restriction for b.

Next in order to derive the equations for T, X, and f, substituting $I(x) = x$ into (3.32) and comparing the coefficients of x^2, we obtain the following equation.

$$T'''(t) = 4\beta^2 T'.$$

[15]The confluent hypergeometirc differential equation is defined as follows:

$$zd^2y/dz^2 + (\mu - z)dy/dz - \lambda y = 0.$$

The above equation is obtained by substituting the transformations $z = \beta x^2/2$, $y = \nu \exp\{-1/2\int^z\{(2\gamma - 1)/s - \beta s d\}s\}$ into the standard confluent hypergeometric function.

$$V(x) = x^{2\nu+1/2} e^{-\beta x^2/4} F(\lambda, \mu; \beta x^2/2).$$

The terms that does not include x is

$$f_0{}'(t) + T''(t)/2 = \gamma T'(t)/4.$$

And $X_0 = 0$.[16] The initial condition leads that $T(0) = 0$ since $\bar{t} = 0$ at $t = 0$, and as to X, $x(0) = x_0$ gives $X(x_0, 0) = 0$, and $f(x_0, 0) = 0$ gives $f_0(0) = x_0{}^2 T''(0)/8$. Solving with respect to T,

$$T(t) = 4\sinh^2 \beta t.$$

Solving with respect to X,

$$X(x,t) = 2\beta x \sinh 2\beta t.$$

As to f,

$$f(x,t) = \gamma \sinh^2 \beta t - (1 + bx)\beta \sinh 2\beta t - x^2\beta^2 \cosh 2\beta t + x_0{}^2\beta^2.$$

Then we can derive the similarity variables using the invariant surface equation (3.30). In case that $T'(t) \neq 0$, solving $dT/T = dX/X$ gives the similarity variable $\xi = x/\sqrt{T}$. The equation for f and solving $dp/p = f dT/T$ lead the functional form of p as follows:

$$\begin{aligned} p(x,t) \quad &= \exp\left[\tfrac{\gamma t}{4} - \tfrac{x_0{}^2\beta}{4}\coth \beta t - \left(\tfrac{\beta x^2}{4}\coth \beta t + \tfrac{1}{2}\int^x b(s)ds\right)\right] \\ &\times T(t)^{1/4}\eta(\xi). \end{aligned}$$

$\eta(\xi)$ is a function of ξ. To determine the functional form of $\eta(\xi)$, these variables should be substituted into the original Fokker-Planck equation. [17] After long calculations, the result gives the following second order ordinal differential equation with respect to η.

$$\eta''(\xi) - \left(\frac{4(2\nu)^2 - 1}{4\xi^2} + x_0{}^2\beta^2\right)\eta = 0.$$

When $x > 0$, using the modified Bessel function $I_{2\nu}(z)$, η can be expressed as follows:

$$\eta(\xi) = \xi^{1/2}\left\{A_1 I_{2\nu}(\beta x_0 \xi) + A_2 I_{-2\nu}(\beta x_0 \xi)\right\}.$$

[16] In case of $T'(t) = 0$, $(b'' - b'b)''' = 0$. This is the equation for b, and it is the same case for substituting $\nu^2 = 1/4/$

[17] Using the similarity variables, p becomes $p(x,t) = D(\xi(x,t),t)\eta(\xi)$. The general Fokker-Planck equation (3.26) is transformed as follows:

$$\begin{aligned} a\left(\tfrac{\partial \xi}{\partial x}\right)^2 \eta'' + \left\{a\left(\tfrac{\partial^2 \xi}{\partial x^2} + \tfrac{2}{D}\tfrac{\partial D}{\partial x}\tfrac{\partial \xi}{\partial x}\right) + b\tfrac{\partial \xi}{\partial x} - \tfrac{\partial \xi}{\partial t}\right\}\eta' \\ + \left\{\tfrac{a}{D}\tfrac{\partial^2 D}{\partial x^2} + \tfrac{b}{D}\tfrac{\partial D}{\partial x} + c - \tfrac{1}{D}\tfrac{\partial D}{\partial t}\right\}\eta = 0 \end{aligned} \quad (3.33)$$

A_1 and A_2 are the integral constants, when $\nu \neq 1/4$, $x > 0$, and $t > 0$, $A_2 = 0$, and

$$A_1 = 2x_0{}^{-2\nu}\,(\beta/2)^{3/4-\nu}\left(F(\lambda,\mu;\frac{\beta x^2}{2})\right)^{-1}.$$

F is the previous Kummer's hypergeometric function. As a special case of this type, substitution of $b(x) = Bx$, and $c(x) = B$ (B is a constant) gives the Ornstein-Uhlenbeck process.

$$a(x)\frac{\partial^2 p(x,t)}{\partial x^2} + (a'(x) + b(x))\frac{\partial p(x,t)}{\partial x} + b'(x)p(x,t) = \frac{\partial p(x,t)}{\partial t}$$ [18]

In case that the diffusion coefficient $a(x)$ is not specified, the solution is obtained when $c(x) = 0$. Feller [1951] calculated a simple case that $a(x) = \alpha x$, and $b(x) = b_1 x + b_0$.

Basically the same procedure is applicalbe as before

$$\begin{aligned} f(x,t) &= f_0(t) + \tfrac{1}{4}\tfrac{dT}{dt} - \tfrac{X_0 J}{2} - \tfrac{1}{4}\tfrac{dT}{dt}IJ - \tfrac{1}{2}\tfrac{dX_0}{dt} - \tfrac{I^2}{8}\tfrac{d^2T}{dt^2} \\ \text{where} \quad J &= \tfrac{da(x)^{1/2}}{dx} + b(x)a^{-1/2}(x) \end{aligned}$$

As to $df_0(t)/dt$, the same procudure produces

$$\begin{aligned} &-\frac{df_0}{dt} - \frac{1}{4}\frac{d^2T}{dt^2} + \frac{1}{2}\frac{d^2X_0}{dt^2}I(x) + \frac{1}{8}\frac{d^3T}{dt^3}I(x)^2 \\ &\qquad = \frac{1dT}{4dt}\left\{\frac{a(x)^{1/2}}{2}\phi'(x)I(x) + \phi(x)\right\} + \frac{X_0(t)}{4}a(x)^{1/2}\phi'(x) \\ &\phi(x) = 2a(x)^{1/2}J'(x) + J(x)^2 - 4b'(x). \end{aligned}$$

I will consider the case $X_0(t) = 0$, but omit the case $X_0 \neq 0$. In case $X_0(t) = 0$, the equation is the quadratic function of $I(x)$ with the coefficients of the functions of t. The twice differential with respect to I leads the constant and separation of variables. Using this relation and $dI/dx = a^{-1/2}$, we obtain the differential equation for ϕ with respect to I.

$$\frac{1}{2}\frac{d\phi(x)}{dI}I(x) + \phi(x) = 2k_1 I(x)^2 + k_2$$

k_1 and k_2 are the integral constants. Integrating this equation gives ϕ. k_3

[18] Hill [1982] pp. 109–115 and Exercises 15–17 (pp. 131–133). I explain some of the expansion.

is also the integral constant.[19]

$$\phi(x) = k_1 I(x)^2 + k_2 + k_3 I(x)^{-2}$$

Substituting this equation into the original df_0/dt and comparing the coefficients of I, we obtain the following equations:

$$\frac{d^3T}{dt^3} - 4k_1\frac{dT}{dt} = 0 \tag{3.34}$$

$$\frac{df_0}{dt} + \frac{1}{4}\frac{d^2T}{dt^2} + \frac{k_2}{4}\frac{dT}{dt} = 0. \tag{3.35}$$

Before integrating these equations and deriving the functional forms of T, X, and f, I will show that the differential equation for ϕ restricts the functional form of J as before. That is, the equation satisfied with the invariant under the infinitesimal transformation (3.28) leads the special relation between the coefficients a and b. Considering I is an independent variable, and expressing $\phi(I) = \phi(x)$, due to $dx = a^{1/2}dI$, we obtain

$$2\frac{dJ}{dI} + J(I)^2 - \frac{4}{a^{1/2}}\frac{db(I)}{dI} = \phi(I).$$

Substituting J's definiton and rearranging it, the equation becomes

$$2\frac{d}{dI}\left\{\frac{d\ln a^{1/2}}{dI} - \frac{b(I)}{a^{1/2}}\right\} + \left\{\frac{d\ln a^{1/2}}{dI} - \frac{b(I)}{a^{1/2}}\right\}^2 = \phi(I).$$

Because this is a Riccati type of the differential equation, let

$$\frac{2}{I}\frac{dV}{dI} = \frac{d\ln a^{1/2}}{dI} - \frac{b(I)}{a^{1/2}} = J - 2\frac{b(I)}{a^{1/2}}.$$

We obtain the following equation:

$$\frac{d^2V}{dV} - \frac{\phi(I)}{4}V = 0.$$

As before, accoriding to the functional form of ϕ, this solution is a Kummer's confluent hypergeometric function.[20]

[19] In case $X_0(t) \neq 0$,

$$\phi(x) = k_1 I(x)^2 + k_2 I(x) + k_4$$

.

[20] In case $X_0 = 0$, $F(\alpha, \gamma; z)$ denotes the Kummer's confluent hypergeometiric function,

Next the initial condition of the equation are given as before

$$p(x,0) = p_0\delta(x - x_0), \quad x(0) = x_0 > 0.$$

If we assume that $I_0 = I(x_0)$ and $T(0) = 0$, the differential equation (3.34) has the three integral constants, and the sum is restricted to zero. Furthremore, $X_0(t) = 0$ means $T'(0) = 0$. Finally, since the remained constant is arbitral, we can choose the coefficient of $T(t)$ is 1. As a result, we obtain

$$T(t) = \sinh^2 \beta t,$$

where $\beta^2 = k_1$.

$$X(t) = I(x)\beta \sinh 2\beta t$$

Equation (3.35) and the initial condition give the following equation:

$$f(x,t) = -\frac{\beta^2 I^2}{4}\cosh 2\beta t - \frac{\beta I J}{4}\sinh 2\beta t - \frac{\beta}{4}\sinh 2\beta t - \frac{k_2}{4}\sinh^2 \beta t + \frac{\beta^2 {I_0}^2}{4}.$$

Using the invariant surface, we can obtain the similarity variable $\xi = I(x)/\sinh \beta t$ and the functional form of p:

$$p(x,t) = \frac{V(I)}{(a \sinh \beta t)^{1/2}} \exp\left[-\frac{k_2}{4} - \frac{\beta}{4}({I_0}^2 + I^2)\cosh \beta t\right]\eta(\xi).$$

The functional form of $\eta(\xi)$ can be obtained by substitution of the variables into the original Fokker-Planck equation.[21]

$$\eta''(\xi) - \left(\frac{\beta^2 {I_0}^2}{4} + \frac{k_3}{4\xi^2}\right)\eta(\xi)$$

This solution is well known as the fisrt type of the modified Bessel function. The transformation

$$z = \beta I_0 \xi/2, \quad \eta(\xi) = \xi^{1/2} y(z)$$

and the coefficients are given as follows:

$$\begin{aligned}
\phi(I) &= k_1 I^2 + k_2 + k_3/I^2 \\
\gamma &= 1 \pm \frac{1}{2}\sqrt{k_3 + 1} \\
\alpha &= \frac{1}{2}\gamma - \frac{k_2}{8{k_1}^{1/2}} \\
z &= {k_1}^{1/2} I^2/2 \\
V(I) &= z^{\gamma/2 - 1/4} e^{-\frac{z}{2}} F(\alpha, \gamma; z).
\end{aligned}$$

[21] We can use the equation (3.33) in the foot note 17.

gives the standard form. Let $\alpha = (1 + k_3)^{1/2}/2$, we obtain the following equation

$$\eta(\xi) = \frac{\eta_0 \xi^{1/2}}{2} \left(I_\alpha(z) + I_{-\alpha}(z) \right).$$

Chapter 4

Stochastic price propagation model

4.1 Price variation Langevin equation: a simple case

In this section, I introduce a stochastic factor in fluctuation of the total factor productivity. It is very difficult to solve the system which incorporates non-linear differential equations and stochastic factors. First, take a single equation and next, consider the two sector system.

4.1.1 One-sector model

The definition of variables are the same as before, the rate of (aggregate) price change is denoted as $d \ln p/dt = z$, a denotes $a = (1 - S_X - S_K)/S_K$, $S_X = p_X x/C$ the cost share for the intermediate input, $S_K = p_L K/C$ the cost share to the capital goods input. $p_K = p(r + \delta - z)$ the user's cost of capital, r the interest rate, δ the depreciation rate.

Assume that the cost shares are constant as in the previous sections. The productivity change $d \ln TFP/dt$ includes a drift term μ_{TFP} and the Langevin term $L(t)$, which is a source of stochastic fluctuation.

Assumption: Total factor productivity (TFP)

$$\frac{d \ln TFP}{dt} = \mu_{TFP} + L(t).$$

Using the above stochastic productivity specification, the one sector model for (1.20) can be expressed as follows:

$$\begin{aligned}
\frac{S_K}{r+\delta-z}\dot{z}+(1-S_X-S_K)z &= -\frac{d\ln TFP}{dt}+S_L\frac{d\ln w}{dt}+\frac{S_K}{r+\delta-z}\dot{r} \\
&\quad +\frac{S_K}{r+\delta-z}\dot{\delta} \\
\dot{z}+a(r+\delta-z)z &= -\frac{r+\delta-z}{S_K}\frac{d\ln TFP}{dt}+\frac{S_L}{S_K}\frac{d\ln w}{dt}+\dot{r}+\dot{\delta} \\
&= -\frac{r+\delta-z}{S_K}(\mu_{TFP}+L(t))+\frac{S_L}{S_K}\frac{d\ln w}{dt} \\
&\quad +\dot{r}+\dot{\delta}
\end{aligned}$$

Assume that the average productivity change μ_{TFP} is allocated into the wage increase, the interest rage change, and the depreciation change, that is

$$\epsilon=\frac{1}{S_K}\left\{S_L\frac{d\ln w}{dt}+S_K(\dot{r}+\dot{\delta})-(r+\delta)\mu_{TFP}\right\},$$

where ϵ may be assumed to be close enough to zero.

Using this equation, rewrite the system as follows:

$$\dot{z}+\left\{a(r+\delta-z)-\frac{\mu_{TFP}}{S_K}\right\}z = -\frac{r+\delta-z}{S_K}L(t)+\epsilon \tag{4.1}$$

Assumption: $L(t)$ the Langevin term Introduce the assumption on the Langevin term:

$$< L(t) >= 0, \quad < L(t)L(t') >= \Gamma\delta(t-t'),$$

where the notation $< \circ >$ denotes taking expectation, Γ denotes variance of $L(t)$, $\delta(t-t')$ denotes Dirac's delta function.

The Langevin equation has multiple interpretations.[1] One of the most famous interpretation is due to Ito, who interprets the time derivative (4.1) as derived from the following integral:

$$\begin{aligned}
z(t+\Delta t)-z(t) &= \int_t^{t+\Delta t}\left\{\frac{\mu_{TFP}}{S_K}-a(r+\delta-z(s))\right\}z(s)+\epsilon ds \\
&\quad -\left\{\frac{r+\delta}{S_K}-\frac{z(t)}{S_K}\right\}\int_t^{t+\Delta t}L(s)ds \\
&= \int_t^{t+\Delta t}\left\{\frac{\mu_{TFP}}{S_K}-a(r+\delta-z(s))\right\}z(s)+\epsilon ds \\
&\quad -\left\{\frac{r+\delta-z(t)}{S_K}\right\}\int_t^{t+\Delta t}L(s)ds.
\end{aligned} \tag{4.2}$$

[1] This part of the chapter is based on van Kampen [1992].

The other famous interpretation is due to Stratonovich, who interprets the time derivative of the left hand side of eq (4.1) as follows:

$$\begin{aligned} z(t+\Delta t) - z(t) &= \int_t^{t+\Delta t} \left\{ \frac{\mu_{TFP}}{S_K} - a(r+\delta - z(s)) \right\} z(s) + \epsilon ds \\ &\quad + \left\{ \frac{r+\delta}{S_K} - \frac{z(t+\Delta t)+z(t)}{2S_K} \right\} \int_t^{t+\Delta t} L(s) ds \\ &= \int_t^{t+\Delta t} \left\{ \frac{\mu_{TFP}}{S_K} - a(r+\delta - z(s)) \right\} z(s) + \epsilon ds \\ &\quad + \left\{ \frac{r+\delta}{S_K} - \frac{z(t+\Delta t)+z(t)}{2S_K} \right\} \int_t^{t+\Delta t} L(s) ds. \end{aligned} \tag{4.3}$$

The interpretation on the coefficient of $L(t)$ differs from each other. Ito's interpretation assumes that z has determined before $L(t)$ has some value, therefore the integral can be taken as if z is a constant. But Stratonovich's interpretation assumes that z moves while $L(t)$ moves into action, therefore the integral has the averaged coefficient. The solution of the system significantly depends on the interpretation taken into account.

Here I assume that a lot of productivity changes occur microscopically, and it is under given macroscopic price variation z. And I also introduce the distribution of $L(t)$ as Gaussian.

Assumption: z Assume no immediate feedback or no reaction from $L(t)$ to z.

Assumption: $L(T)$ Assume $L(t)$ obeys Gaussian process.

Under these assumptions, the Ito's integral (4.4) is equivalent to the Fokker-Planck equation (4.5).

$$z(t+\Delta t) - z(t) = \int_t^{t+\Delta t} A(z(s)) ds + C(z(t)) \int_t^{t+\Delta t} L(s) ds \tag{4.4}$$

$$\frac{\partial P(z,t)}{\partial t} = -\frac{\partial \left(A(z)P(z,t)\right)}{\partial z} + \frac{\Gamma}{2} \frac{\partial^2 \left(C(z)^2 P(z,t)\right)}{\partial z^2} \tag{4.5}$$

This equation can be expanded as follows:

$$\begin{aligned} \frac{\partial P(z,t)}{\partial t} &= \left\{ -A'(z) + \Gamma \left(C''C + C'^2 \right) \right\} P(z,t) \\ &\quad + (2\Gamma CC' - A) \frac{\partial P(z,t)}{\partial z} + \frac{\Gamma C^2}{2} \frac{\partial^2 P(z,t)}{\partial z^2} \end{aligned} \tag{4.6}$$

Introduce new parameters α and β as follows:

$$\begin{aligned}
&\alpha(z) = \frac{\Gamma C(z)^2}{2}\\
&\alpha'(z) = \Gamma C(z)C'(z)\\
&\beta(z) = \Gamma C(z)C'(z) - A(z)\\
&\beta'(z) = \Gamma(C'(z)^2 + C(z)C''(z) - A'(z))\\
&\beta(z) + \alpha'(z) = 2\Gamma C(z)C'(z) - A(z).
\end{aligned} \tag{4.7}$$

Equation (4.6) is rewritten to

$$\begin{aligned}
&\frac{\partial P(z,t)}{\partial t} = \beta'(z)P(z,t) + (\alpha'(z) + \beta(z))\frac{\partial P(z,t)}{\partial z} + \alpha(z)\frac{\partial^2 P(z,t)}{\partial z^2}\\
&\frac{\partial P(z,t)}{\partial t} = \frac{\partial(\beta(z)P(z,t))}{\partial z} + \frac{\partial\left\{\alpha(z)\frac{\partial P(z,t)}{\partial z}\right\}}{\partial z}
\end{aligned} \tag{4.8}$$

Apply these relations to (4.2), and the following equations are obtained:

$$\begin{aligned}
&A(z) = -\left\{a(r+\delta-z) - \frac{\mu_{TFP}}{S_K}\right\}z + \epsilon\\
&C(z) = -\frac{r+\delta-z}{S_K}\\
&\alpha(z) = \frac{\Gamma}{2}\left(\frac{r+\delta-z}{S_K}\right)^2\\
&\alpha'(z) = -\Gamma\frac{r+\delta-z}{S_K{}^2}\\
&\beta(z) = (r+\delta-z)\left\{az - \frac{\Gamma}{S_K{}^2}\right\} - \frac{\mu_{TFP}}{S_K}z - \epsilon\\
&\beta'(z) = \frac{\Gamma}{S_K{}^2} + a(r+\delta) - 2az - \frac{\mu_{TFP}}{S_K}\\
&\beta(z) + \alpha'(z) = (r+\delta-z)\left\{az - 2\frac{\Gamma}{S_K{}^2}\right\} - \frac{\mu_{TFP}}{S_K}z - \epsilon
\end{aligned} \tag{4.9}$$

Thus, equation (4.2) is equivalent to the following Fokker-Planck equation.

$$\begin{aligned}
\frac{\partial P(z,t)}{\partial t} \quad = \quad & \frac{\partial\left\{\left((r+\delta-z)(az-\Gamma/S_K{}^2)-\mu_{TFP}/S_K-\epsilon\right)P(z,t)\right\}}{\partial z}\\
& + \frac{\Gamma}{2S_K{}^2}\frac{\partial\left\{(r+\delta-z)^2\partial P(z,t)/\partial z\right\}}{\partial z}.
\end{aligned}$$

Notice that $P(z,t)$ is different from $p(t)$.

The solution for this equation can be obtained by numerical calculation. Strict solution has been found only for very specific cases (for example Rogers [1997]).[2]

In the Langevin equation for price fluctuation, the price's diffusion coefficient (instantaneous variance) is proprotional to the ratio of real interest

[2]Finding strict solutions for the Fokker-Planck equation see for example, Nariboli [1977], Hill [1982], and Zwillinger [1998]

rate over the capital's cost share, and the drift coefficient is reciprocal to the variacne of thechnical progress Γ. But in the steady state, the real interest rate is zero, and the price's diffusion coefficient has no effect on the prices in spite of technical change. At the other steady state of no inflation rate $z = 0$, the system is unstable. If the price changes to negative, it contiues to infinity. If there is no fluctuation effect (the Langevin term), the price does not change. But if the system incorporates a random effect, the Langevin term becomes a shock for deflationary spiral. From the diagrams in Chapter 2, there is less possibility for inflationary process. Once starting the spiral, the original small fluctuation has no significant effect on the prices.

4.2 Stochastic formulation for the n-sector model

In the multiple sector model, introduce $\boldsymbol{\epsilon}$ for unallocated productivity gain:

$$\begin{aligned}\boldsymbol{\epsilon} &= \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix} \\ &= \begin{pmatrix} \mu_{TFP\,1} - \sum_j^{n_L} S_{Lj1} \frac{d \ln w_j}{dt} - \sum_{i=1}^{n} \frac{S_{K\,i1}}{r_1 + \delta_{i1} - z_i} \dot{r}_1 \\ - \sum_{i=1}^{n} \frac{S_{K\,i1}}{r_1 + \delta_{i1} - z_i} \dot{\delta}_{i1} \\ \mu_{TFP\,2} - \sum_j^{n_L} S_{Lj2} \frac{d \ln w_j}{dt} - \sum_{i=1}^{n} \frac{S_{K\,i2}}{r_2 + \delta_{i2} - z_i} \dot{r}_2 \\ - \sum_{i=1}^{n} \frac{S_{K\,i2}}{r_2 + \delta_{i2} - z_i} \dot{\delta}_{i2} \\ \vdots \\ \mu_{TFP\,2} - \sum_j^{n_L} S_{Lj2} \frac{d \ln w_j}{dt} - \sum_{i=1}^{n} \frac{S_{K\,i2}}{r_2 + \delta_{i2} - z_i} \dot{r}_2 \\ - \sum_{i=1}^{n} \frac{S_{K\,i2}}{r_2 + \delta_{i2} - z_i} \dot{\delta}_{i2} \end{pmatrix}.\end{aligned} \tag{4.10}$$

$$\frac{d \ln \mathbf{TFP}}{dt} = \begin{pmatrix} \frac{d \ln TFP_1}{dt} \\ \frac{d \ln TFP_2}{dt} \\ \vdots \\ \frac{d \ln TFP_n}{dt} \end{pmatrix} = \begin{pmatrix} \mu_{TFP\,1} + L_1(t) \\ \mu_{TFP\,2} + L_2(t) \\ \vdots \\ \mu_{TFP\,n} + L_n(t) \end{pmatrix} \tag{4.11}$$

In general, the dynamic price equation system can be expressed introducing $\boldsymbol{\epsilon}$, the Langevin term $\mathbf{L}(\mathbf{t})$ and $\boldsymbol{z}$ as follows:

$$\mathbf{S r \dot{z}} + (\mathbf{I} - \mathbf{S_X} - \mathbf{S_K})\boldsymbol{z} = \mathbf{L} + \boldsymbol{\epsilon} \tag{4.12}$$

4.2.1 The system around the fixed point z=0

First, investigate the system around the fixed point $z = 0$,

$$\mathbf{Sr}(0)\dot{z} + (\mathbf{I} - \mathbf{S_X} - \mathbf{S_K})z = \mathbf{L} + \epsilon. \quad (4.13)$$

In this case, the system can be expressed as

$$dz(t) = -\mathbf{Sr_0}^{-1}\{(\mathbf{I} - \mathbf{S_X} - \mathbf{S_K})z(t) - \epsilon\}dt + \mathbf{Sr_0}^{-1}d\mathbf{L}(t), \quad (4.14)$$

where $\mathbf{Sr_0} = \mathbf{Sr}(0)$.

To solve the equation, take the following autonomous matrix equation first,[3]

$$dz(t) = -\mathbf{Sr_0}^{-1}\{(\mathbf{I} - \mathbf{S_X} - \mathbf{S_K})z(t)\}dt$$

It has a solution $\psi(t, t_0)$ corresponded to an initial condition.

The non-autonomous equations can be rewrite as follows:

$$dz(t) = -\mathbf{Sr_0}^{-1}\{(\mathbf{I} - \mathbf{S_X} - \mathbf{S_K})z(t) - \epsilon\}dt + \mathbf{Sr_0}^{-1}d\mathbf{L}(t).$$

Using the solution $\psi(t, t_0)$, the solution for the non-autonomous equations are given as follows:

$$z(t) = \psi(t, t_0)\left(z(t_0) + \int_{t_0}^{t} \psi(s)^{-1}\epsilon ds + \int_{t_0}^{t} \psi(s, t_0)^{-1}\mathbf{Sr_0}^{-1}d\mathbf{L}(s)\right)$$

Stability of the system depends on the eigenvalues of the matrix

$$-\mathbf{Sr_0}^{-1}(\mathbf{I} - \mathbf{S_X} - \mathbf{S_K}).$$

The condition is the same as that of the non-stochastic model that I have explained.

The solution method of the non-linear stochastic system has not yet been investigated. But the Fokker-Planck equations for multiple variables can be determined from eq (4.12). And there are many numerical methods to solve the non-linear partial differential equations, see for example Zwillinger [1998].

Assume $\mathbf{L(t)}$ obeys the Gaussian process and has the covariances $< L_i(t)L_j(s) >= 2\Gamma_{ij}\delta(t-s)$, eq (4.12) which become equivalent to the multivariate Fokker-Planck equation. Thus the Fokker-Planck equations for eq (4.12) can be shown as follows:

[3]Following part of this section depends on Hori [1977].

$$\begin{aligned}\frac{\partial P(\mathbf{z},t)}{\partial t} &= -\sum_{j=1}^{n}\frac{\partial}{\partial z_j}\left(\left(\mathbf{Sr}^{-1}(\mathbf{I}-\mathbf{S_X}-\mathbf{S_K})\mathbf{z}-\boldsymbol{\epsilon}\right)P(\mathbf{z},t)\right)\\ &\quad +\frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{\partial^2}{\partial z_i\partial z_j}\boldsymbol{\Omega}_{ij}P(\mathbf{z},t)\right),\end{aligned}$$

where

$$\boldsymbol{\Omega} = 2\mathbf{Sr}^{-1}(\mathbf{z})\Gamma{\mathbf{Sr}^{-1}}^{T}(\mathbf{z}).$$

${\mathbf{Sr}^{-1}}^{T}$ denotes transpose of the matrix, Γ denotes $n\times n$ matrix of the covariance of $\mathbf{L}(t)$. These equations remains to be investigated.

Chapter 5

Concluding remarks

This research has first examined the dynamic properties of inter–industry wage and productivity changes. Even in the very simplest case, output price fluctuations differ greatly, depending on whether the model includes capital price formation or not. Our model generates the capital gains or losses so that we can properly describe price fluctuations in case of high technological developments and relatively low wage increases. Furthermore we considered the case where there is a wide difference in productivity growth between the sectors. The decrease in output prices in the high TFP sector will eventually lowers the output prices for the relatively low TFP (and relatively high wage) growth sector. But in the path to this final situation the rental price of capital, which is produced in the high productivity sector, will increase, because of the capital loss effects. This capital loss makes the output prices increase. Thus this output price increase will result in the capital gains effects, which in turn decrease the cost of capital. These interactions can be easily shown in the simple two-sector model.

Our numerical examples shows the dynamic properties of inter-industry productivity growth incorporating capital gains. One of the properties is a cyclical pattern of price variation, which brings the transmission process to a steady state. Another property is divergence of price changes.

Next I constructs the classification for the general two sector model based on the price equations for growth accounting. As a result, the degree of out-souring on capital goods is an important factor for the stability of the system. If each sector out-sources their capital goods for investment extensively, the fixed point of the system becomes a saddle point, rather than the asymptotically stable point.

But as the phase portraits show, the system includes a large instability area, which is separated by the locus of singularity of the system's matrix $\mathbf{Sr}$. When the prices come across this singularity locus, the fluctuation explodes drastically. This is why my previous paper that the numerical changes in prices very fragile.

As for a stochastic model, there are a lot of problems to be solved. The non-linear stochastic equation system is extremely difficult to solve, but it is necessary to derive the distribution of prices, that will explain how frequent the price changes go into an unstable region, such as a deflation spiral, or hyper inflation. For that purpose, I have derived the general partial differential equation system for future investigation.

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